Chapter 1

Electro Magneto-statics : Highlights

The following presentations were designed, drawn and edited by Motti Deutsch (0544 767 567, motti.jsc@gmail.com).

- Mathematics will be teached in respect to the physical problem.
- Exercises: questions and solutions will be set in the course's site.
- There is no obligation to submit solved exercises.
- Three exams: 45 minutes each, questions taken from exercises (closed material) (30%).
- Test, 3 hours, 3 out of 4 question should be shoved (open material) (70%).

- Bibliography:

1. Umran & Aziz S. Inan: Engineering Electromagnetics

2. D.J. Griffiths : Introduction to Electrodynamics

3. Berkley: Electricity & Magnetism

4. J. D. Jackson: Classical Electrodynamics (Third Edition)

5. Alonso – Finn, Fundamental University Physics:

Mechanics I: Chapters 6 and 11
Fields and Waves: Chapter 17

Coulomb's law
Gauss Theorem
Divergence Theorem

Units

$$MKS \rightarrow m, Kg, sec. \quad ; \quad \mathbf{cgs} \rightarrow cm, gr., sec.$$

$$F = ma \rightarrow \{Kg \cdot m \cdot sec^{-2}\} \equiv 1\mathbf{Newton}$$:MKS

$$1\text{N} = 10^3 gr 10^2 cm \cdot sec^{-2} =$$ $: cgs$

$$=10^5 gr \cdot cm \cdot sec^{-2} \equiv 10^5 dyn$$

Charges

cgs: when two equal charges, 1cm apart, act upon each other by a force of 1 dyne the charge is defined as 1 electrostatic unit – esu or statcoulomb.

MKS: **Coulomb** (C - International System). It is the charge carried by a constant current of one ampere in one second.

$$1C = 3 \cdot 10^9 esu$$

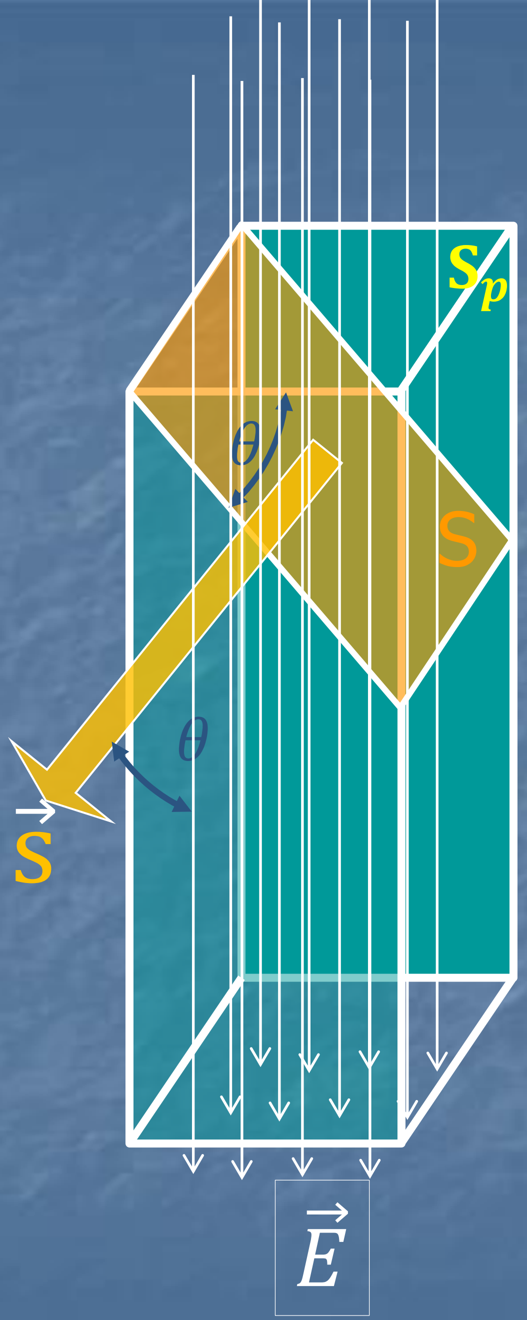
Coefficients:

$$K_e\{MKS\} = 9 \cdot 10^9 \mathrm{N} m^2 C^{-2}$$

Given that: $1C = 3 \cdot 10^9 esu \Rightarrow$

$$\Rightarrow K_e\{cgs\} =$$
$$= 9 \cdot 10^9 \cdot 10^5\, dyn \cdot 10^4 cm^2 (3 \cdot 10^9 esu)^{-2} =$$

$$= \mathbf{1}\{dyn \cdot cm^2 \cdot esu\}$$



The general expression for line of flux

$p \rightarrow perpendicular$ $\vec{S} \rightarrow$ Surface vector

$$\phi_e = ES_p = ES\cos\theta = \vec{E}\cdot\vec{S}$$

Where E the **force per unit charge** and/or the **flow density**.
When E is space-dependent then:

$$d\phi_e = \vec{E}\cdot\overrightarrow{ds}$$

The total flow lines through **S**:

$$\phi_e = \int_S \vec{E}\cdot\overrightarrow{ds}$$

The gradient (1)

The work per unit charge is $\frac{\Delta W_e}{q} = \int_{r_1}^{r_2} \vec{E} \cdot \vec{dl} = -[V(r_2) - V(r_1)] = -\int_{r_1}^{r_2} dV = \int_{r_1}^{r_2} \left(-\frac{\partial V}{\partial l}\right) dl$

Follow the yellow-highlights:

$$\int_{r_1}^{r_2} \vec{E} \cdot \hat{l}\, dl = \int_{r_1}^{r_2} E\cos\theta dl = \int_{r_1}^{r_2} E_T dl$$

$$\vec{E} \cdot \hat{l} = E\cos\theta = -\frac{\partial V}{\partial l} \equiv -\frac{\partial V}{\partial \vec{l}}$$

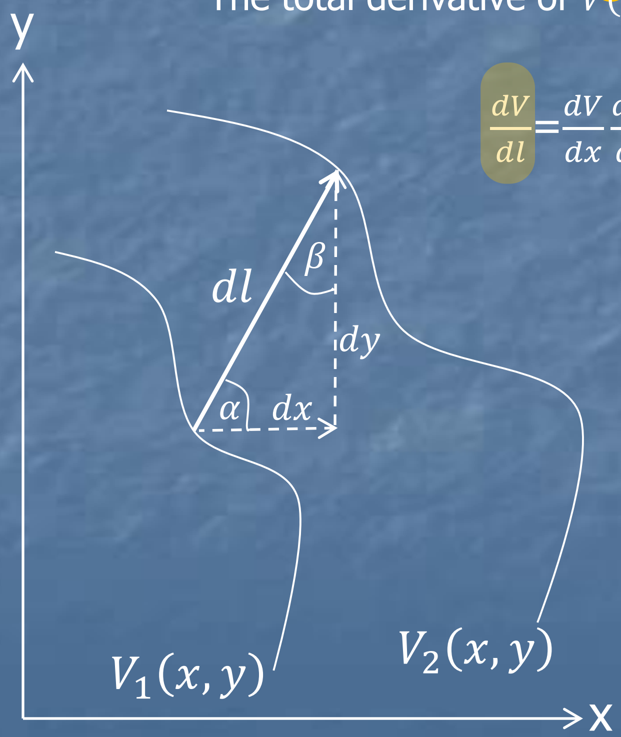
directional derivative

and therefore:

$$\Rightarrow E = \left|\frac{\partial V}{\partial \vec{l}}\right|_{\max}$$

The gradient essence

The total derivative of $V(x, y)$ with respect to dl:

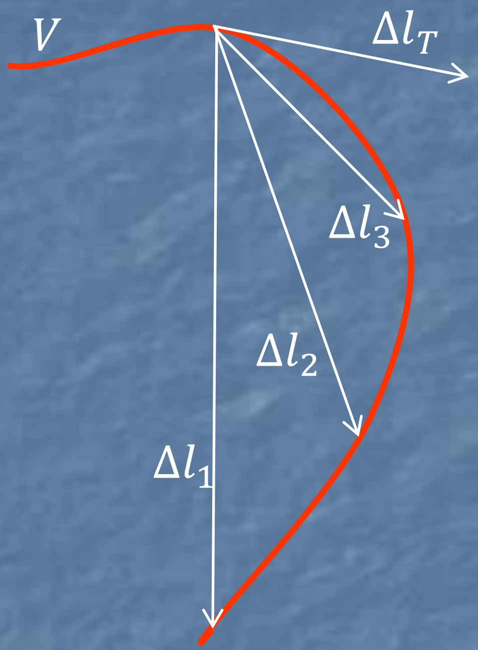


$$\frac{dV}{dl} = \frac{dV}{dx}\frac{dx}{dl} + \frac{dV}{dy}\frac{dy}{dl} = \frac{dV}{dx}\cos\alpha + \frac{dV}{dy}\cos\beta =$$

$$\frac{dV}{dx}\hat{x} \cdot \hat{l} + \frac{dV}{dy}\hat{y} \cdot \hat{l} = \left(\frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y}\right) V(x,y) \cdot \hat{l} \equiv \overrightarrow{grad\, V} \cdot \hat{l}$$

$$\vec{E} = -\overrightarrow{gradV}$$

The gradient (2): direction



$$\frac{\Delta V}{\Delta l_1} = \frac{\Delta V}{\Delta l_2} = \frac{\Delta V}{\Delta l_3} = 0 \Rightarrow$$

$$0 = \lim_{\Delta l \to 0} \frac{\Delta V}{\Delta l} = \frac{dV}{dl_T} = \overrightarrow{gradV} \cdot \hat{u}_T = 0$$

$$\Rightarrow \vec{E} = -\overrightarrow{gradV} \perp \hat{u}_T$$

The electric field lines (flux) is always perpendicular to an equipotential surface and hence work done in moving a charge between two points on an equipotential surface is zero.

Some other relations:

$$1.\quad \phi_e = \oint_S \vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0} \quad ; \quad \vec{\nabla}\vec{E} \equiv div\vec{E} = \frac{\rho}{\varepsilon_0} \quad ;$$

$$2.\quad \Delta V_{cl.tr} = \oint_L \vec{E}\cdot d\vec{l} = 0 \quad ; \quad ??? \qquad \vec{E} = -\vec{\nabla}V = -gradV$$

Combining the right expressions of 1 and 2 yield the del squared (Laplacian) ∇^2 :

$$\vec{\nabla}\vec{E} = \vec{\nabla}(-\vec{\nabla}V) = -divgradV = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

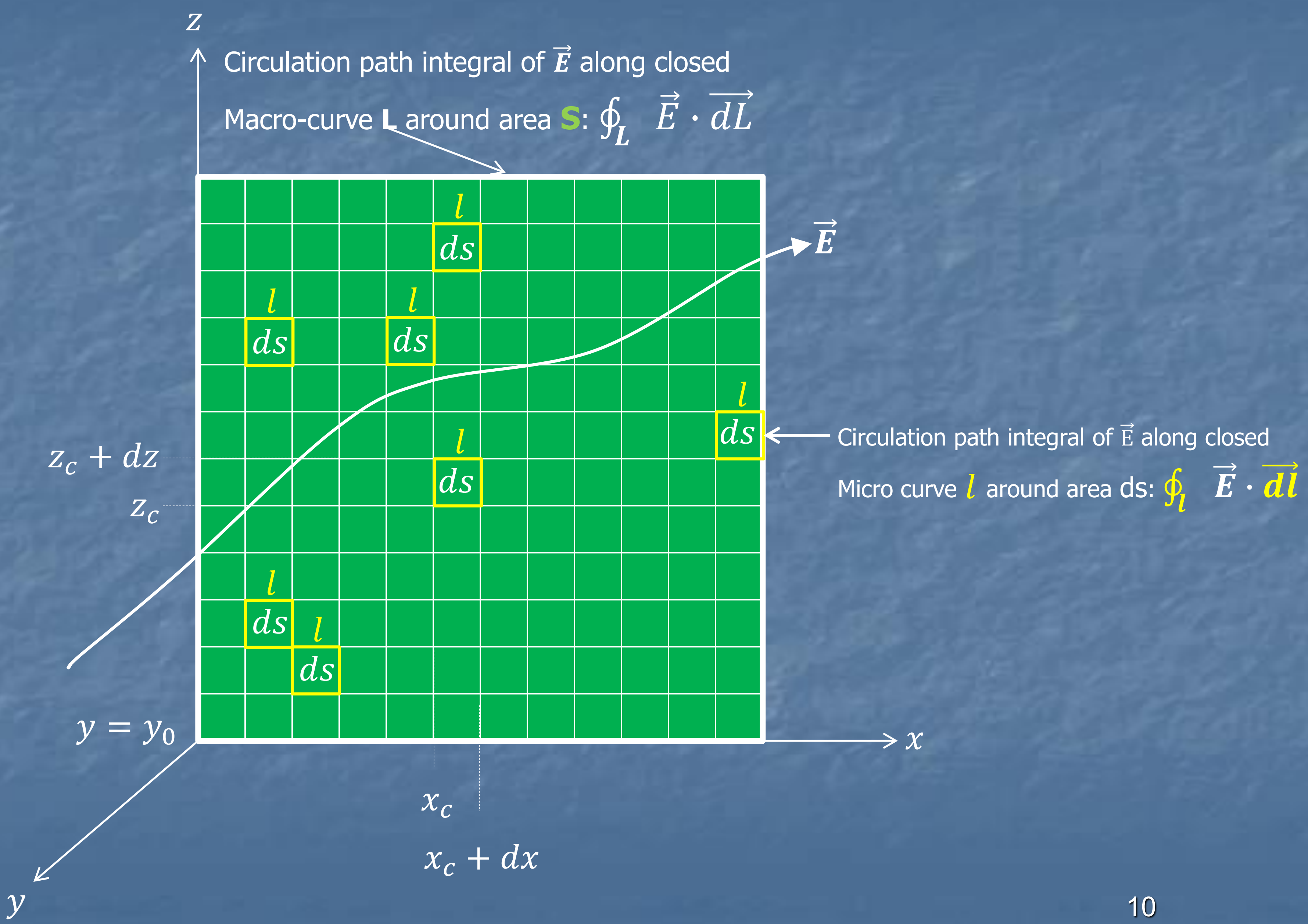
$$\nabla^2 V = \frac{\rho}{\varepsilon_0} \quad Poisson's\ equation$$

$$\nabla^2 V = 0 \quad Laplace's\ equation$$

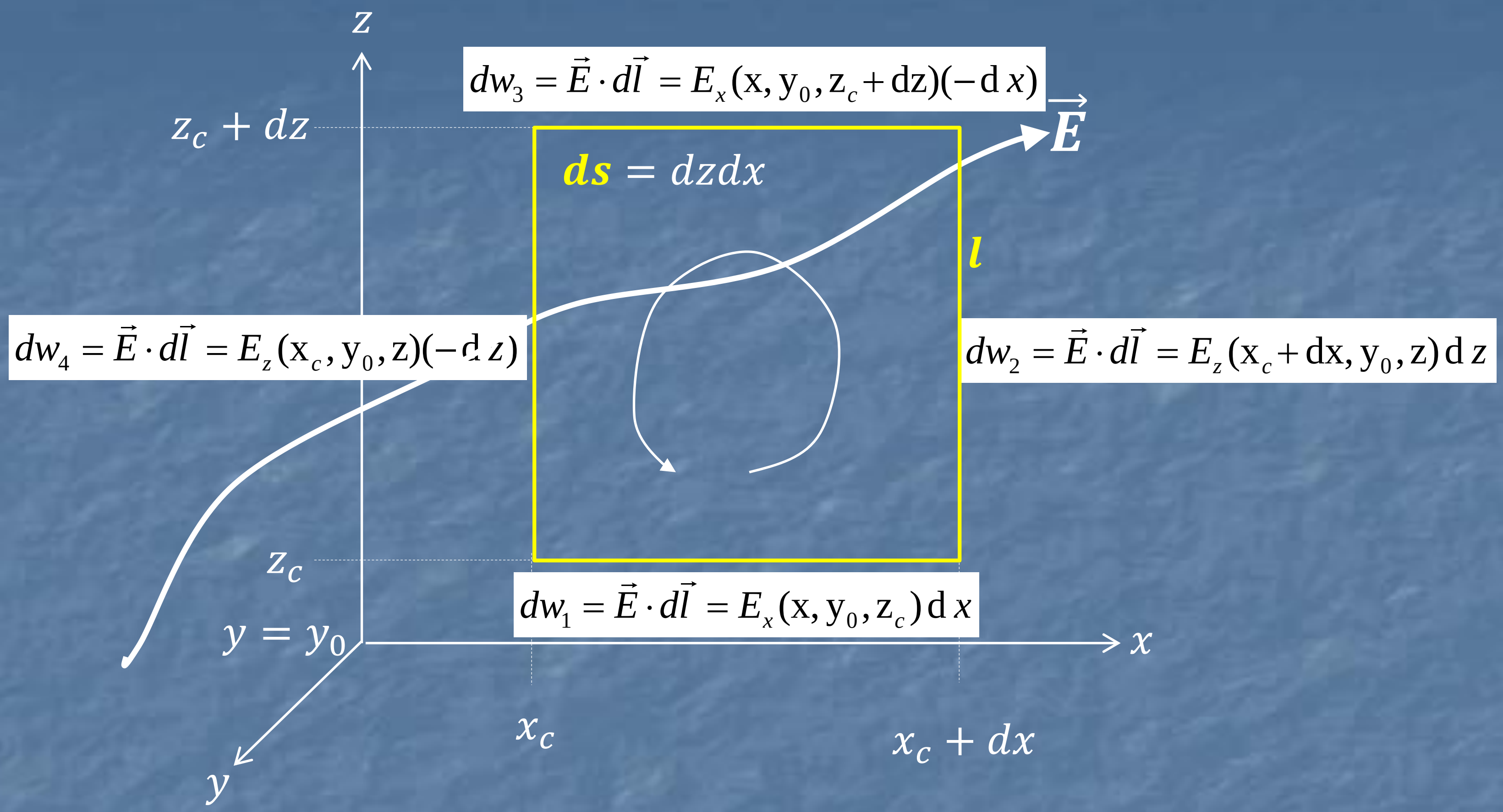
Hence, $\vec{E}$ can be calculated via:

1. Coulomb's law
2. Gauss's law
3. The potential: $\vec{E} = -\overrightarrow{gradV}$ (reason for potential continuity)

(1) The differential aspect of the circulation theorem - The Rotor



(2) The circulation integral of $\vec{E}$ over infinitesimal path l (line integral) which defines the area (region) ds



And the work done along ***l*:**

$$\oint_{\text{Infinitesimal } l\ (ds)} \vec{E}\cdot d\vec{l} = \underbrace{\left[E_z(x_c+dx, y_0, z) - E_z(x_c, y_0, z)\right]dz}_{\left[\frac{\partial E_z}{\partial x}dx\right]dz} - \underbrace{\left[E_x(x, y_0, z_c+dz) - E_x(x, y_0, z_c)\right]dx}_{\left[\frac{\partial E_x}{\partial z}dz\right]dx}$$

(3) Line integral over infinitesimal path

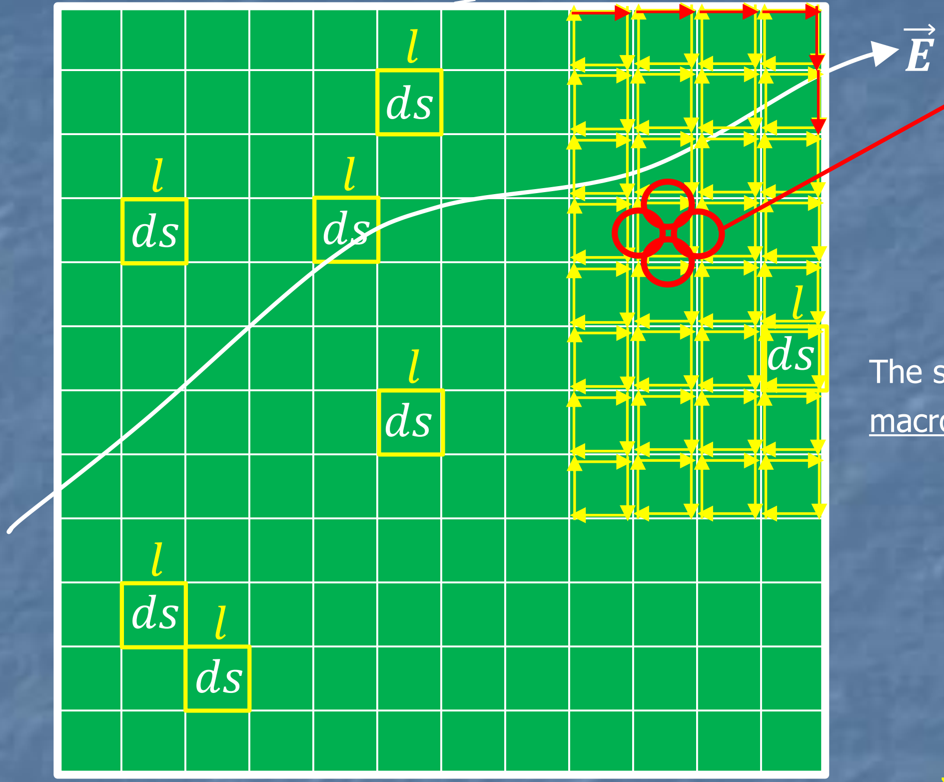
$$\oint_{x-z\ plane} \vec{E}\cdot d\vec{l} = \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) dxdz = \left(\vec{\nabla} X \vec{E}\right)_y ds_y = \left(\vec{\nabla} X \vec{E}\right)\cdot ds_y \hat{y}$$

And in general, regarding line integral around infinitesimal region ds which has a three-component projection on x-y, x-z, and y-z planes:

$$\oint_{l(ds)} \vec{E}\cdot d\vec{l} = \left(\vec{\nabla} X \vec{E}\right)\cdot d\vec{s}$$

Stokes' theorem

Circulation path integral of $\vec{E}$ along closed Macro-curve **L** around area **S**: $\oint_L \vec{E}\cdot\overrightarrow{dL}$



$$\vec{E}\cdot(-\vec{dl})+\vec{E}\cdot(+\vec{dl})=0$$

Circulation path integral of $\vec{E}$ along closed Micro curve l around area ds:

$$\oint_l \vec{E}\cdot\overrightarrow{dl}=\nabla X\vec{E}\cdot d\vec{s}\begin{cases} =0, within\ S \\ \neq 0\ on\ the\ circumference\ L\end{cases}$$

The sum of the private microscopic circulation integrals (along l_i *around* $d\vec{s}_i$) over the entire macro plane S, arrested by the macroscopic path L is:

$$\sum_{l_i}^{S}\int_l \vec{E}\cdot\overrightarrow{dl}=\sum_{d\vec{s}}^{S}(\nabla X\vec{E})_i\cdot d\vec{s}=\oint_L \vec{E}\cdot\overrightarrow{dL}$$

$$\lim_{d\vec{s}\to 0}\sum_{d\vec{s}}^{S}(\nabla X\vec{E})_i\cdot d\vec{s}=\int_S \nabla X\vec{E}\cdot d\vec{s}=\oint_L \vec{E}\cdot\overrightarrow{dL}$$

Stokes' theorem

Hence, the line integral per unit area is: $\frac{\partial}{ds}\left(\oint_L \vec{E}\cdot\overrightarrow{dL}\right)=\nabla X\vec{E}$

For conservative electric vector field: $\oint_{L(S)} \vec{E} \cdot \overrightarrow{dl} = 0$

Hence, according to Stokes' theorem: $0 = \oint_{L(S)} \vec{E} \cdot \overrightarrow{dl} = \int_{S(L)} \nabla X \vec{E} \cdot d\vec{s}$

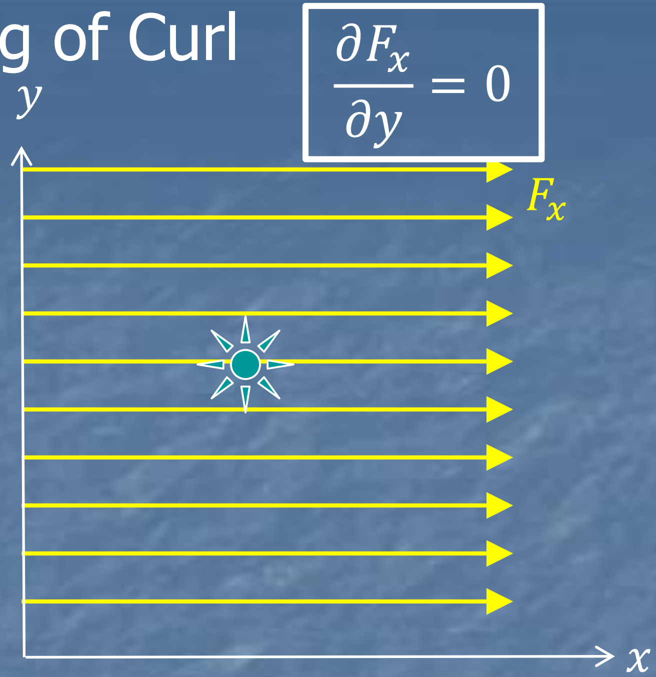
And therefore: $\nabla X \vec{E} = 0$

All the electrostatic fields are conservative:

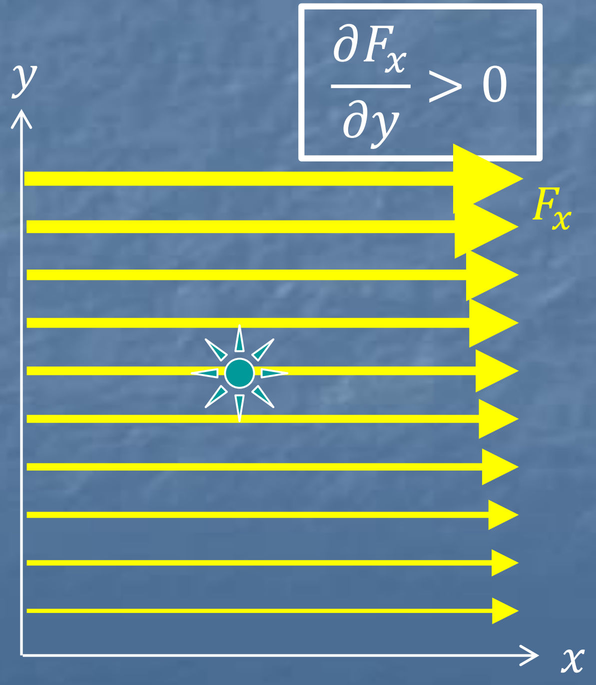
$$\vec{E} = K\sum_{i=1}^{n}\frac{q_i}{r_i^2}\hat{r}_i = K\sum_{i=1}^{n}(-grad\,\frac{q_i}{r_i}) \Rightarrow \vec{\nabla}\,\mathrm{X}\,\vec{E} = K\,curl\left[\sum_{i=1}^{n}(-grad\,\frac{q_i}{r_i})\right] = K\left[\sum_{i=1}^{n}(-curlgrad\,\frac{q_i}{r_i})\right] = 0$$

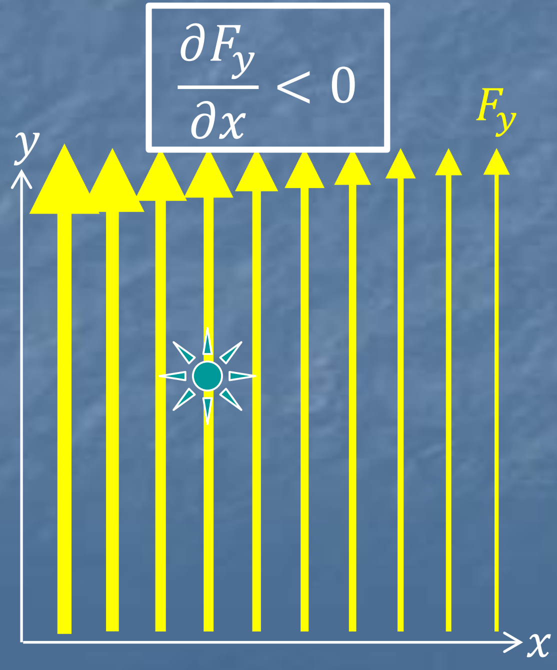
$$curlgrad = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} = \hat{x}\underbrace{\left(\frac{\partial^2}{\partial y\partial z} - \frac{\partial^2}{\partial z\partial y}\right)}_{=0} + \hat{y}\underbrace{\left(\frac{\partial^2}{\partial z\partial x} - \frac{\partial^2}{\partial x\partial z}\right)}_{=0} + \hat{x}\underbrace{\left(\frac{\partial^2}{\partial x\partial y} - \frac{\partial^2}{\partial y\partial x}\right)}_{=0} = 0$$

The meaning of Curl



- A flow is traveling in the $x - y\ pane\ towards\ \hat{x}$
- Within the flow, a torque testing profiled wheel with teeth (גלגל שיניים) is placed. The flow acts upon the wheel circumference by a torque of the force F_x.
- Where does torque exist? (upper/lower panels)
- How this can be mathematically expressed?
- Does any other fundamental heterogeneous situation exist?

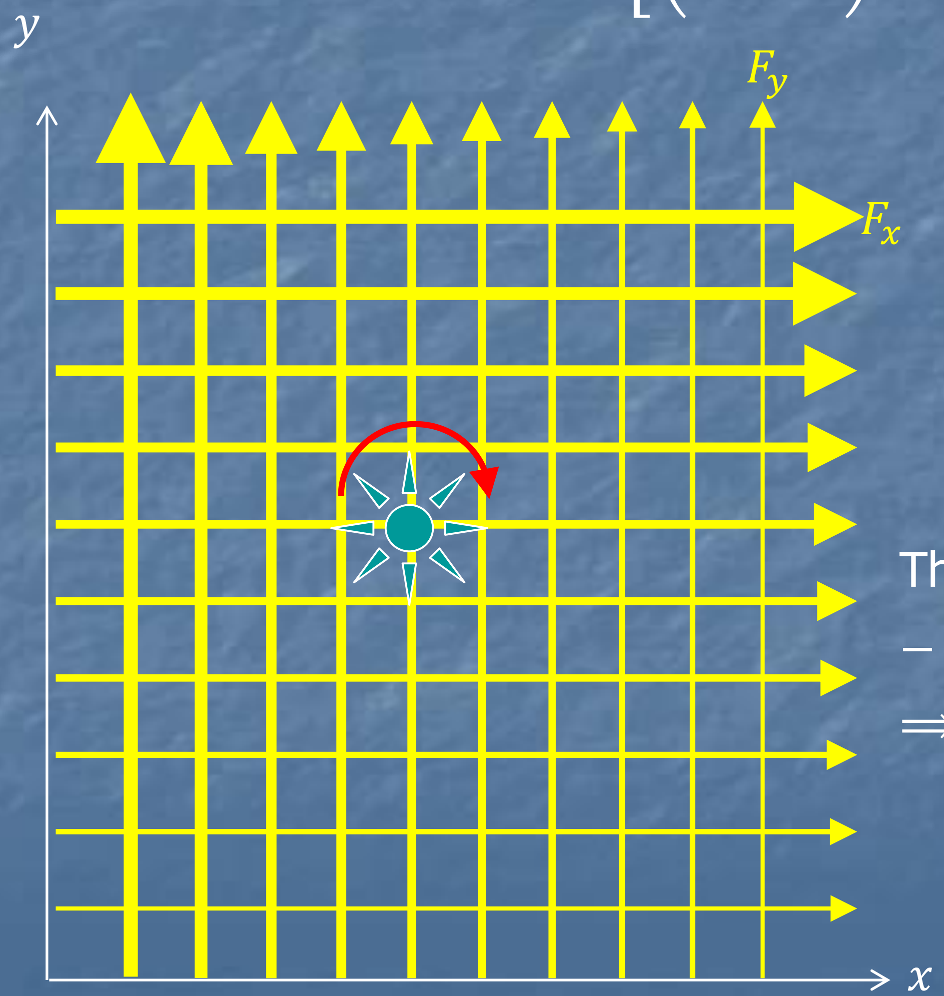


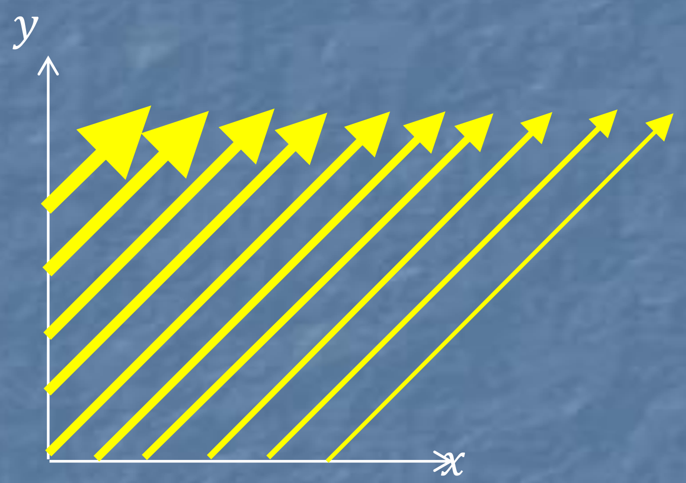


for the general case in $x - y$ plan, regarding our example:

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \equiv \left(\nabla X\vec{F}\right)_z \neq 0 \text{ since} \rightarrow$$

$$\rightarrow \left(\nabla X\vec{F}\right)_z = \left[\left(-\frac{\partial F_y}{\partial x}\right) - \frac{\partial F_x}{\partial y}\right]\hat{z} = -\left(\frac{\partial F_y}{\partial x} + \frac{\partial F_x}{\partial y}\right)\hat{z} \neq 0$$

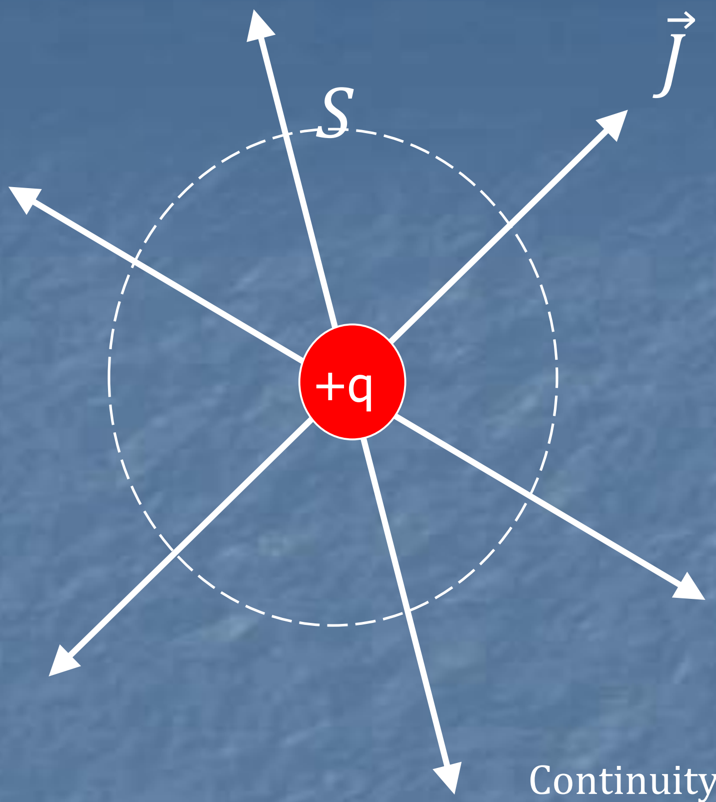




Therefore, in the present example the direction of $\left(\nabla X\vec{F}\right)_z$ is $-\hat{z}$ which presents a clockwise rotation and since $\left(\nabla X\vec{F}\right)_z \neq 0$ $\Longrightarrow$ $\vec{F}$ is non-conservative.

Examples

continuity equation



$$\Longrightarrow \quad \oint \vec{J} \cdot d\vec{s} = -\frac{dq}{dt}$$

Divergence theorem

$$\int div\, J\, dV = -\frac{\partial}{\partial t}\int \rho dV \;\Rightarrow\; div\, J = -\frac{\partial \rho}{\partial t} \;\Rightarrow$$

Continuity equation via ρ:

$$div\, J + \frac{\partial \rho}{\partial t} = 0$$

But: $div\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \rho = \varepsilon_0 div\vec{E}$

and introducing it into the above equation yield the expression of the continuity equation via the electric field.

$$\Longrightarrow div\, \vec{J} + \frac{\partial(\varepsilon\, div\vec{E})}{\partial t} = 0 \qquad \Longrightarrow div\,(\vec{J} + \frac{\partial(\varepsilon\,\vec{E})}{\partial t}) = div\,(\bar{J}_C + \frac{\partial(\bar{D})}{\partial t}) = div\,(\bar{J}_C + \bar{J}_D) = 0$$

Conducting and displacement currents

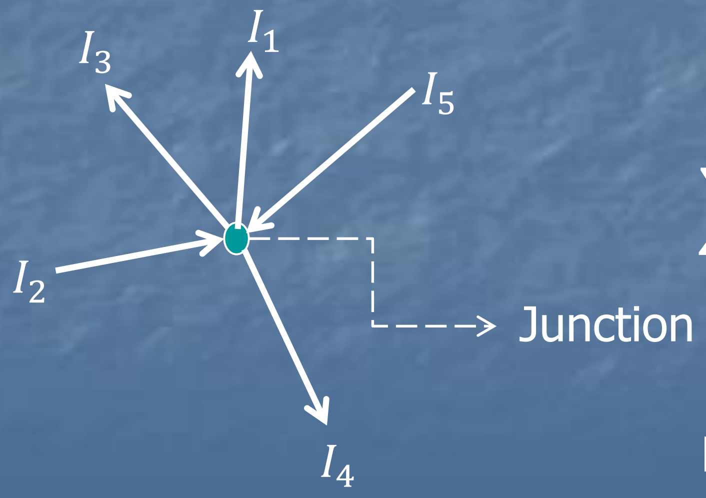
Kirchhoff's current law (**junction rule**)

We get: $$div\ J + \frac{\partial \rho}{\partial t} = 0$$

In steady state $\frac{\partial \rho}{\partial t} = 0$

$div\ J = 0 \longrightarrow$ divergence $\longrightarrow 0 = \int div\ J\ dV = \oint \vec{J} \cdot d\vec{s} = 0$

The right integral teaches that the sum of the currents through the closed shell is zero. The practical meaning of that is that (regarding electric circuits) the sum of currents in (through) a junction must be zero (at zero or at very low frequency).



$$\sum_j I_j = 0$$ **Kirchhoff's junction rule**

Explain the significance of $\frac{\partial \rho}{\partial t} \neq 0$

Relaxation time of the charge density in conductor having conductivity σ and permittivity ε_0

$$J = \sigma E \qquad divE = \rho/\varepsilon_0$$

$$divJ = div(\sigma E) = \sigma divE = \sigma \frac{\rho}{\varepsilon_0} = -\frac{d\rho}{dt}$$

$$\int_{\rho_0}^{\rho_t} \frac{d\rho}{\rho} = -\int_0^t \frac{\sigma}{\varepsilon_0} dt \Rightarrow \rho(t) = \rho_0 e^{-\frac{t}{\varepsilon_0/\sigma}} = \rho_0 e^{-\frac{t}{\tau}} \quad ; \quad \frac{\varepsilon_0}{\sigma} \equiv \tau \to RC = \frac{l}{\sigma s}\frac{\varepsilon_0 s}{l} = \frac{\varepsilon_0}{\sigma}$$

Silver: $\sigma = 6.2 \text{ x } 10^7 Sm^{-1}$; $\varepsilon \cong \varepsilon_0 = 8.85 \text{ x } 10^{-12} N^{-1} m^{-2} C^2$

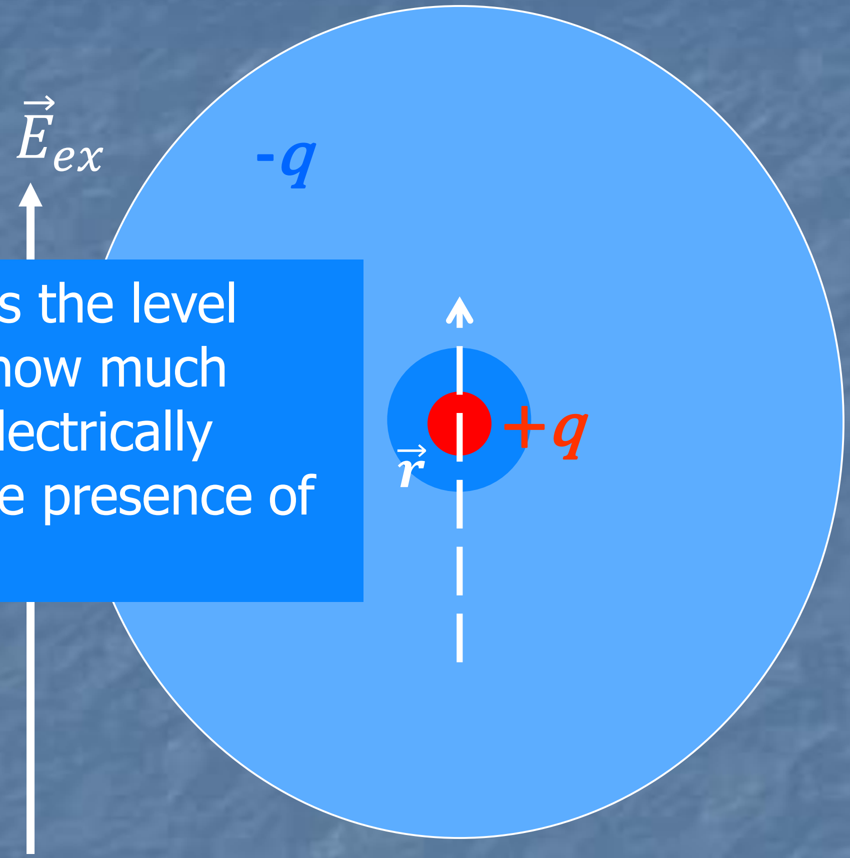
$$\tau_{Silver} = \frac{8.85 \text{ x } 10^{-12}}{6.2 \text{ X } 10^7} \cong 10^{-19} sec$$

Insulators:

$$\tau^{DDW} \cong 10^{-5} \text{ sec}$$
$$\tau^{anber} \cong 4 \cdot 10^3 \text{ sec}$$
$$\tau^{mica} \cong 10 - 20\, hours$$
$$\tau^{quartz} \cong 50\, days$$

Insulators

Have no free charges. In the presence of electric field they are electrically polarized. The resultant electric field/charge within the insulator is never zero.



Susceptibility is the level (measure) of how much substance is electrically polarized in the presence of electric field.

- In the figure the centers of the positive and negative charges are coinsized
- An electric field acts upon the atom upwards
- As a result, mostly the negative cloud-charge moves downwards while the positive nuclei upwards.
- Now, the distance between the centers is $\vec{r}$
- Though subjected to electric field, it is weak enough to keep both the negative cloud shape and distribution (symmetry) spherical, and the density constant, $\rho = \rho_0$.
- Hence, the electric field from the negative charge at $\vec{r}$ i:

$$\vec{E}_P(\vec{r}) = \frac{\rho_0 \vec{r}}{3\varepsilon_0} = \frac{\frac{-q}{4\pi a^3/3}\vec{r}}{3\varepsilon_0} = \frac{-q}{4\pi a^3 \varepsilon_0}\vec{r} = -\vec{E}_{ex}(\vec{r}) \Rightarrow$$

$\equiv \alpha$

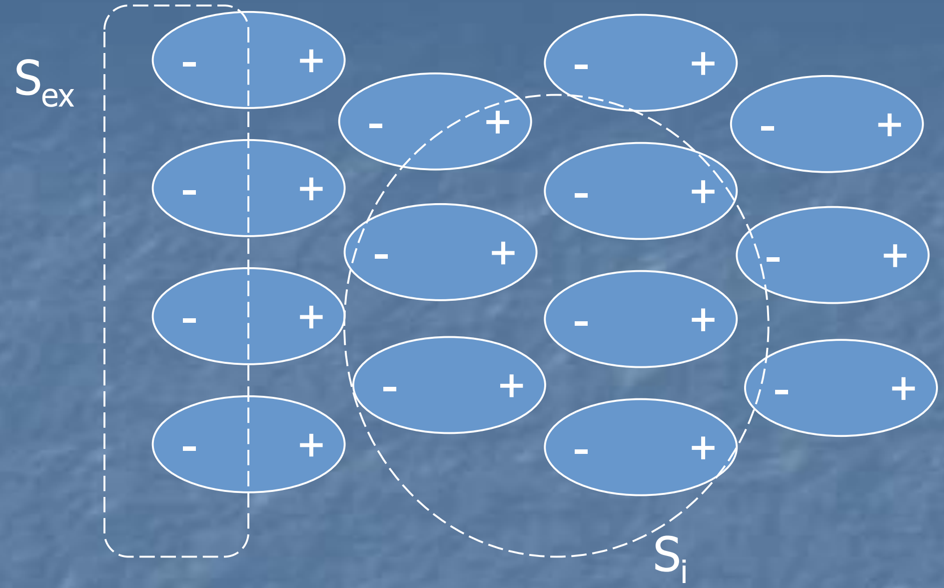
$$\Rightarrow 4\pi a^3 \cdot \varepsilon_0 \cdot \vec{E}_{ex}(\vec{r}) = q\vec{r} \equiv \vec{p}$$

This relation is sufficiently correct (by factor 4) for simple atoms. $4\pi a^3$ is defined as the atomic contribution and called the atomic polarizability α. Next, if in a dielectric unit volume there exist n atoms, then the polarization per unit volume $\vec{P}$ of that substance is:

$$\vec{P} = \varepsilon_0 n\alpha \vec{E}_{ex} = \varepsilon_0 \chi_e \vec{E}_{ex} \;\; ; \quad \chi_e \equiv Electric\ susceptibility$$

The electric field both induces and orients the electric dipole.

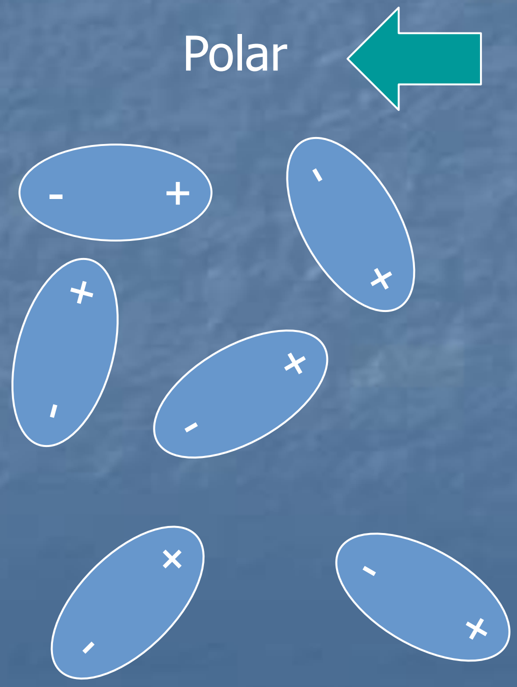
Nonpolar

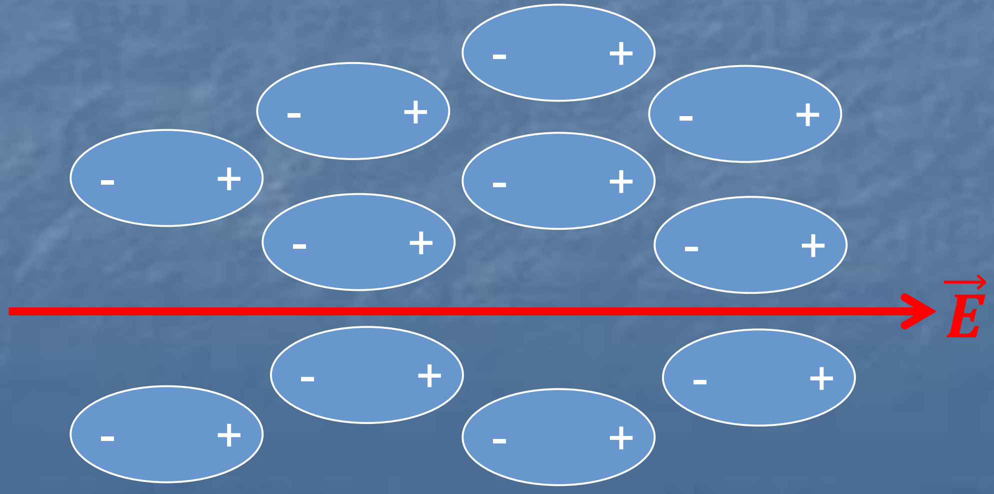


(a)

Polar

Within the dialectic, over macroscopic volumes (S_i), the net charge is zero, while on the surface of this volume (S_{ex}) a net charge is accumulated (induced) and causes the electric dipole of the entire substance.





(b)

Units of $\vec{P}$, χ_e, and the displacement vector $\vec{D}$:

$$\{\vec{P}\} = \left\{\frac{P}{m^3}\right\} = \left\{\frac{q\vec{r}}{m^3}\right\} = \frac{Cm}{m^3} = \frac{C}{m^2} \equiv \textit{surface charge density}\,(\text{SCD})$$

From Gauss's law:

$$\oint_S \vec{E}\cdot d\vec{s} = \frac{q}{\varepsilon_0} \Rightarrow \oint_S \varepsilon_0\vec{E}\cdot d\vec{s} = q \Rightarrow \left\{\varepsilon_0\vec{E} \equiv \vec{D}\right\} = Cm^{-2} \;\; ; \text{SCD}$$

But we show that:

$$\vec{P}\left\{Cm^{-2}\right\} = \varepsilon_0\chi_e\vec{E}_{ex} = \chi_e\left(\varepsilon_0\vec{E}_{ex}\left\{Cm^{-2}\right\}\right) \Rightarrow$$

$$\Rightarrow \left\{\chi_e\right\} = \textit{Dimensionless quantity}$$

The displacement vector $\vec{D}$ is defined as:

$$\varepsilon_0\vec{E} \equiv \vec{D} \;\; (\textbf{displacement vector})$$