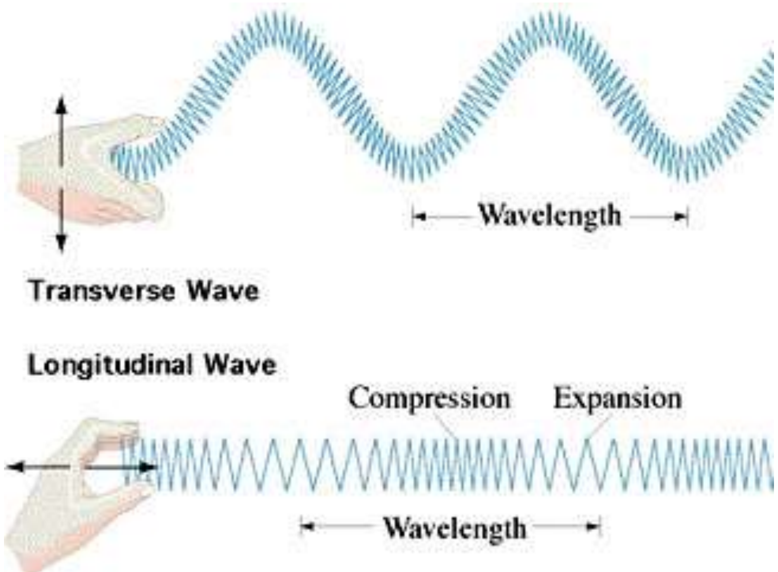
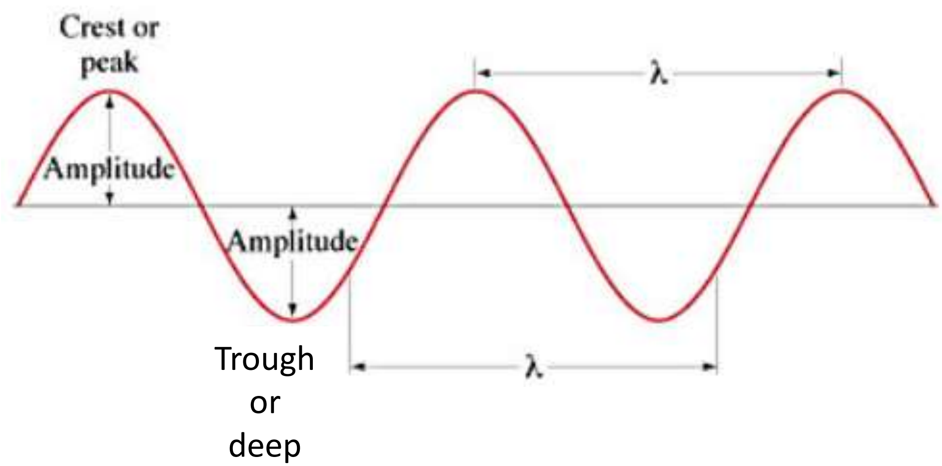
Chapter 2: EM Waves and properties

Waves, a reminder

Wave characteristics (1):



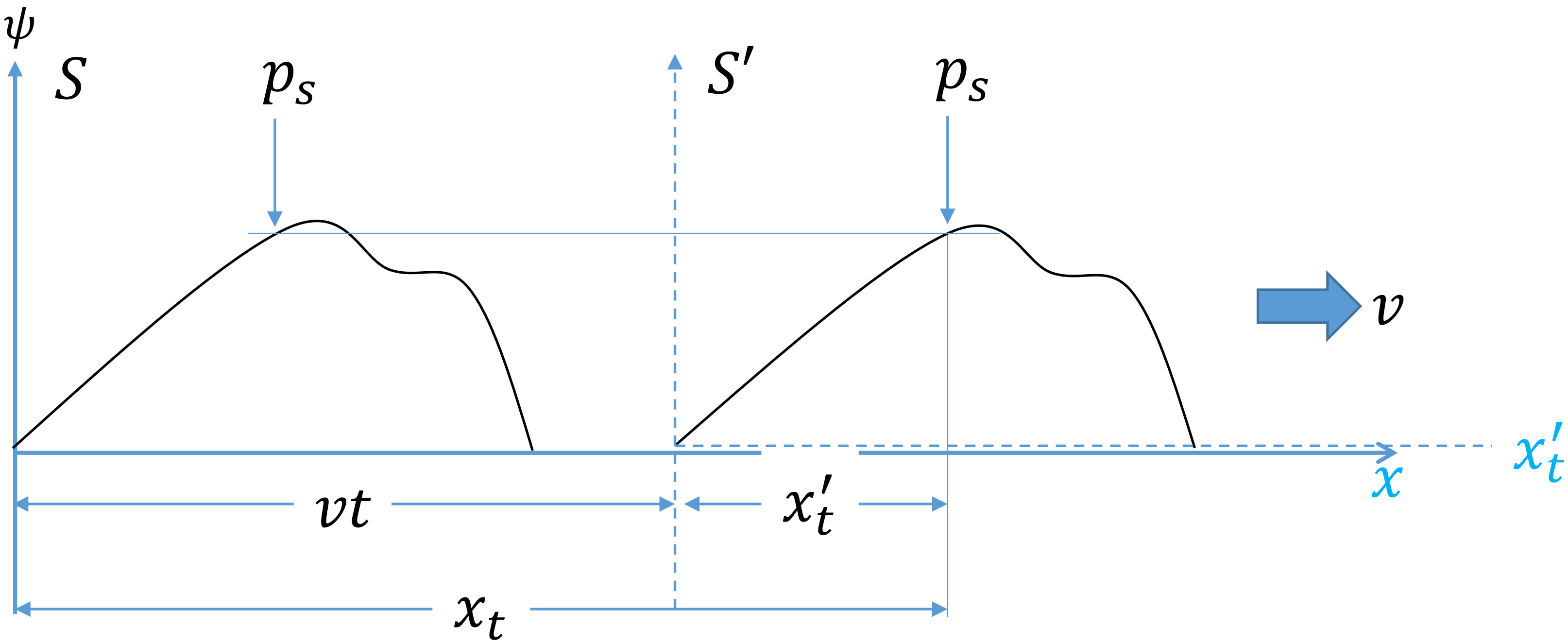


Wave is a disturbance propagating though a medium. The disturbance moves, but the medium itself does not. Physical waves come in two varieties: **transverse** and **longitudinal** and are characterized by three parameters: **amplitude, frequency,** and **wavelength**.

Both varieties are described by the same periodic function $\psi(x,t) = f(x \pm vt)$, where v is the propagation velocity of the wave (disturbance/perturbation).

https://phet.colorado.edu/en/simulations/category/physics/sound-and-waves

Wave characteristics (2):



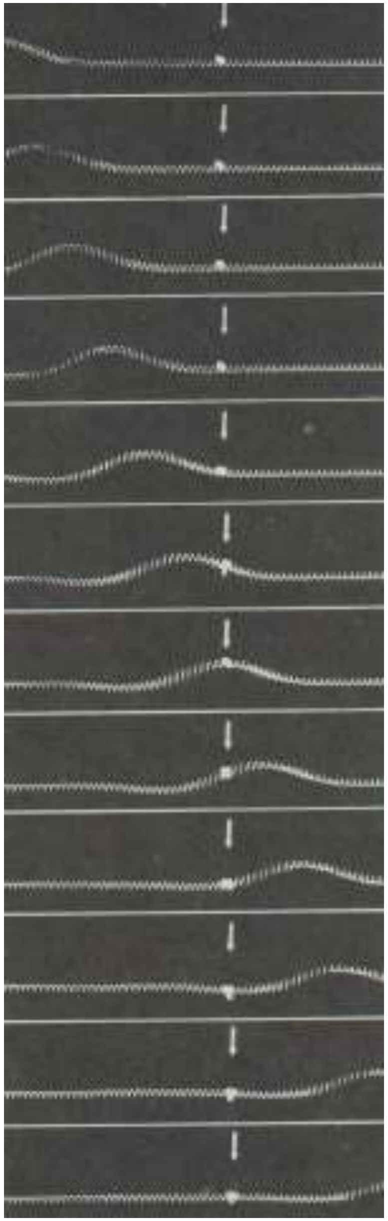
The disturbance ψ (its nature is not important) which moves in the positive x direction with a constant velocity v, and must be a function of both positions x and time t and can be expressed as $\psi = \mathrm{f}(\mathrm{x}, \mathrm{t})$. The shape of ψ at any instance, say at $t = 0$, can be found by holding time constant at that value, i.e.:

$$\psi = \mathrm{f}(\mathrm{x}, \mathrm{t})|_{t=0} = \mathrm{f}(\mathrm{x}, 0) = \mathrm{f}(\mathrm{x})$$

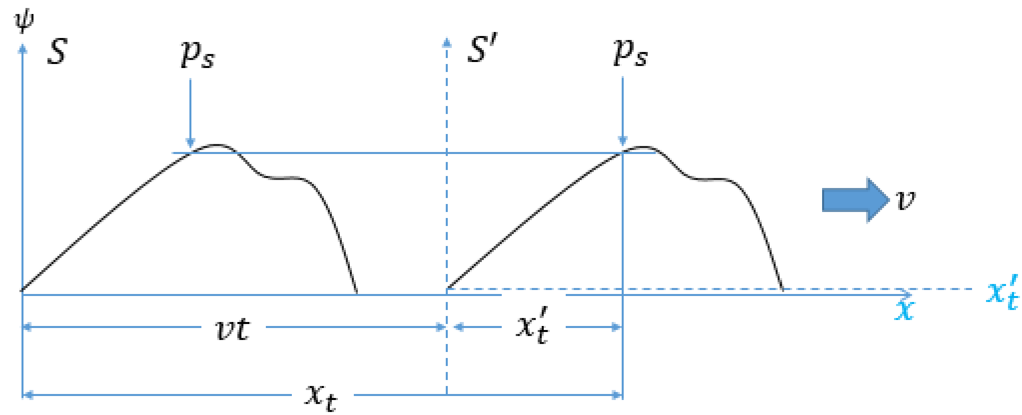
Representing the shape or profile of the wave at that time point.

In a way, the process is analogous to “taking” a “photograph” of the disturbance as it travels by.

Propagation of a pulse on a spring. The section of the spring moves up and down as the pulse travels from left to right.



Wave characteristics (3):



Relating to ψ (not deformed through space) in coordinate system S', which travels with the pulse at a speed v, ψ is no longer a function of time, and as we move along with S', we see a stationary constant profile with the same functional form as at $t = 0$, i.e. $\psi = f(\mathrm{x}, \mathrm{t})|_{t=0} = f(\mathrm{x}, 0) = f(\mathrm{x})$ but for x', $\psi = f(\mathrm{x}')$

According to the figure above, $\mathrm{x}' = \mathrm{x}_t - vt$. Hence ψ can be written in terms of the variables associated with the stationary S system as:

$$\psi(x, t) = f(x - vt)$$

This then, represents the most general form of one-dimensional wave function. It should be emphasized that one should only choose the shape (function), say j(x), and then substitute $(x - vt)$ for x in j(x), to make it a wave.

Hence, for instance, $\psi(x, t) = \psi_0 e^{-(x-vt)^2}$ is a bell-shaped wave, traveling in the positive x direction with a speed v.

Wave characteristics (4):

Does $\psi(x,t) = f(x \pm vt) = \mathrm{f(u)}$ indeed addresses the partial wave function?

To prove that, we apply the chain rule for derivation [i.e. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$]. In our case:

$$\frac{\partial u}{\partial x} = 1 \qquad \text{and} \qquad \frac{\partial u}{\partial t} = \pm v.$$

Then: $$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial \psi}{\partial u} \qquad \text{and} \qquad \frac{\partial \psi}{\partial t} = \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial t} = \pm v\frac{\partial \psi}{\partial u}$$

Next, taking the second derivatives, one gets:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{d}{du}\left(\frac{\partial \psi}{\partial x}\right)\frac{\partial u}{\partial x} = \frac{d^2 \psi}{du^2} \text{[a]} \quad \textit{and} \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{d}{du}\left(\frac{\partial \psi}{\partial t}\right)\frac{\partial u}{\partial t} = \pm v \cdot \frac{\partial^2 \psi}{\partial u^2} \cdot \pm v = v^{\,2}\frac{\partial^2 \psi}{\partial u^2} \quad \text{[b]}$$

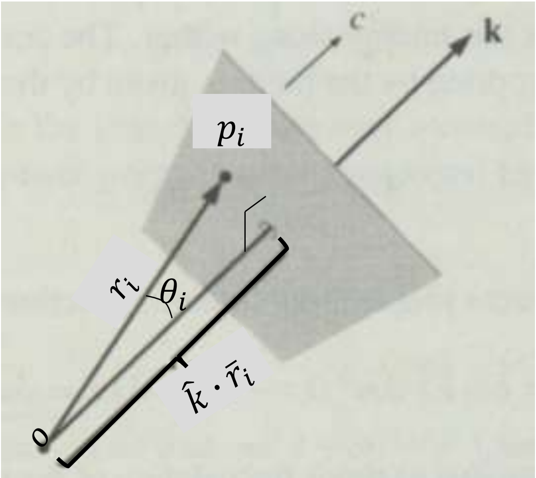
Combining both results, a and b, to eliminate $\frac{\partial^2 \psi}{\partial u^2}$, we get:

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^{\,2}}\frac{\partial^2 \psi}{\partial t^2}$$

Proving that $\psi(x,t) = f(x \pm vt)$ is a solution of the partial differential wave equation, independent of the form of the function f.

Wave characteristics (5): The vector significance of the wave number $\bar{k}$

A wave front (upper figure) is a surface on which points p_i are affected in the same way by a wave at a given time.

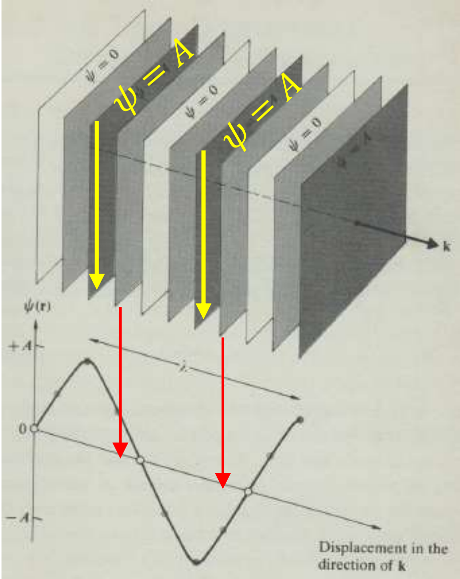


Plane wavefront, $\bar{\boldsymbol{k}}$, $\bar{\boldsymbol{r}}_i$ and $\hat{\boldsymbol{k}}\cdot\bar{\boldsymbol{r}} =$ **constant**

In other words: the surface on which points have identical phases, i.e. $\phi = \left(\bar{k}\cdot\bar{r} \pm \omega t\right) = constant$ [like potential surface in electrostatics] for a given time point.

The shortest form of the equation of a plane perpendicular to $\bar{\boldsymbol{k}}$ is: $\bar{\boldsymbol{k}}\cdot\bar{\boldsymbol{r}} =$ constant, i.e.

The product $\bar{k}\cdot\bar{r}_i = krcos\theta_i$ must be the same for every $\bar{r}_i$ ($\bar{k}$ $is\ constant$).

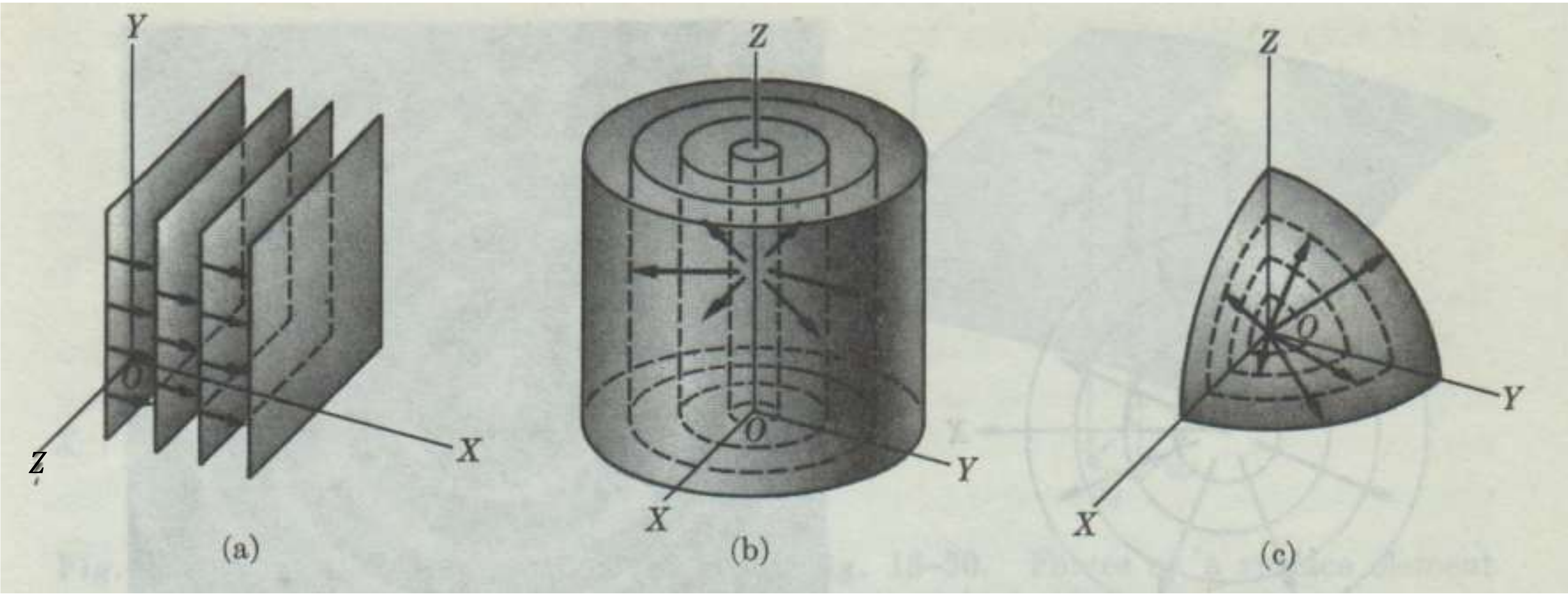


Wavefront for a harmonic plane wave

This yields a plane to which $\bar{k}$ must be perpendicular to and $\bar{r}_i$ is the position of each point p_i on this plane in respect to a given origin).

A representative wave $\psi(r)$ and corresponding plane wave fronts are given in the lower figure.

Wave characteristics (6): Various shapes of waveforms



Wave fronts: plane (a), cylindrical (b) and spherical (c). A-F 699

1. Plane electromagnetic wave in an unbound medium

1.1 Plane Wave in a Simple, Source-Free $\left(\rho,\ \bar{J}, \bar{p}, \overline{\mathcal{M}} = 0\right)$ and Lossless Medium

Where ρ is the volume density of free net charge, $\bar{J}$ is the current surface density, $\bar{p}$ is the polarization vector in dielectric (Coulombs· m^{-2}) and $\overline{\mathcal{M}}$ is the volume Magnetization density vector (Ampere· m^{-1}) in magnetic medium.

Our starting point consists of the 4 differential Maxwell equations:

1.1.1 $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$

1.1.2 $\nabla \cdot \bar{E} = -\frac{\rho}{\varepsilon_0}$

1.1.3 $\nabla \times \bar{B} = -\mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$

1.1.4 $\nabla \cdot \bar{B} = 0$

The two curl Maxwell's equations indicate the fact that changing Magnetic Field with time (1.1.1) produces Electric Fields and vice versa (1.1.3) and hence, necessarily lead to propagation of electromagnetic waves.

1.1.1 $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$

1.1.3 $\nabla \times \overline{B} = -\mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}$

Taking the curl of 1.1.1 and substituting it into the right side of 1.1.3 gives:

$$\nabla \times \nabla \times \overline{E} = -\frac{\partial}{\partial t} \nabla \times \overline{B} = -\frac{\partial}{\partial t}\left(-\mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2}$$

Vector analysis teaches that $\nabla \times \nabla \times \overline{E} = \nabla(\nabla \cdot \overline{E}) + \nabla^2 \overline{E} = \nabla^2 \overline{E}$ since $\nabla \cdot \overline{E} = \rho = 0$ one gets:

$$\nabla^2 \overline{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{E}}{\partial t^2} \qquad [2]$$

Similarly, taking the curl of 1.1.3 and substituting Eq. 1.1.1 into it's right side gives:

$$\nabla \times \nabla \times \overline{B} = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\nabla \times \overline{E}) = \rightarrow Eq. 1.1.1 = -\mu_0 \varepsilon_0 \frac{\partial}{\partial t}\left(-\frac{\partial \overline{B}}{\partial t}\right) = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{B}}{\partial t^2}$$

Again, $\nabla \times \nabla \times \overline{B} = \nabla(\nabla \cdot \overline{B}) + \nabla^2 \overline{B} = \nabla^2 \overline{B}$ and, since $\nabla \cdot \overline{B} = 0$, one gets:

$$\nabla^2 \overline{B} = \mu_0 \varepsilon_0 \frac{\partial^2 \overline{B}}{\partial t^2} \qquad [3]$$

We identify Equations 2 and 3 as differential wave equations for $\overline{E}$ and for $\overline{B}$

Hence in Vacuum, Maxwell Equations teach~~es~~ that each of the Cartesian (scalar) components of $\overline{E}$ and $\overline{B}$ obeys the **three-dimensional wave equation**:

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Further more, Maxwell Equations imply that:

1. electromagnetic waves indeed propagate in vacuum and
2. these waves travel~~s~~ at the speed c:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{1.2566 \cdot 10^{-6} m \cdot kg \cdot C^{-2} \cdot 8.8541878 \cdot 10^{-12} N^{-1}(kg^{-1} \cdot m^{-1} \cdot \sec^2) m^{-2} C^2}} =$$

$$= 0.29979 \cdot 10^9 \frac{m}{\sec} \approx 3 \cdot 10^8 \frac{m}{\sec} = 300{,}000 \frac{km}{\sec}$$

Maxwell's speculation : "This velocity is so nearly that of light (measured by Fizeau (1849): **313,300 km/second**, M.D.), that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiation~~s~~ if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws."

History intermezzo:

Hippolyte Fizeau 1819-1896

In 1849, Fizeau calculated a value for the speed of light to a better precision than the previous value determined by Ole Romer in 1676. He used a beam of light reflected from a mirror 8 kilometers away. The beam passed through the gaps between teeth of a rapidly rotating wheel. The speed of the wheel was increased until the returning light passed through the next gap and could be seen.

James Clerk Maxwell 1831–1879

Maxwell: “This velocity is so nearly that of light, that it seems we have strong reason to conclude that light itself (including radiant heat, and other radiations if any) is an electromagnetic disturbance in the form of waves propagated through the electromagnetic field according to electromagnetic laws.”

Einstein on Maxwell’s work: “most profound and the most fruitful that physics has experienced since the time of Newton”.

Partial differential equations, a reminder:

Many of the problems of mathematical physics involve the solution of partial differential equations. In electromagnetics, these can be generally divided into two types of second order partial differential equations:

Laplace's equation: $\nabla^2 u = 0$, where the function ***u*** *might describe the gravitational/electrical potential* functions *in no-matter/charge region and steady state temperature in a non-heat source region as well.*

Poisson's equation: $\nabla^2 u = f(x,y,z)$, where ***u*** may present the same physical quantities listed for **Laplace's equation**, but in regions containing matter/electric charges, etc. The function $f(x,y,z)$ is called 'the source density', for instance in electricity it is related to ρ_e .

Diffusion of heat flow equation: $\nabla^2 u = \frac{1}{\alpha^2}\frac{\partial u}{\partial t}$, where ***u*** may present non-steady state temperature in a non-heat source region or the concentration of diffusing material. α is a constant which is defined as the *diffusivity.*

Wave equation: $\nabla^2 u = \frac{1}{v^2}\frac{\partial^2 u}{\partial^2 t}$, where ***u*** may present the displacement from equilibrium of (a) vibrating string/membrane, or (in acoustics) the vibrating medium (gas, liquid, solid), of (b) the electrical current or potential along a transmission line and of (c) the components $\overline{E}$ and $\overline{B}$ of an electromagnetic wave.

We will analyze the case of the differential wave equation for $\overline{E}$ (Equation 2). First, let us write the full expression:
(Equation 2):

$$\nabla^2 \overline{E} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x(x,y,z,t)\hat{x} + \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_y(x,y,z,t)\hat{y} +$$

$$+ \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_z(x,y,z,t)\hat{z} =$$

$$= \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \left(E_x(x,y,z,t)\hat{x} + E_y(x,y,z,t)\hat{y} + E_z(x,y,z,t)\hat{z} \right)$$

So the wave equation independently holds true for each of the components of the vector field $\overline{E}$.
For convenience, we shall solve the scalar wave equation for E_X:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x(x,y,z,t) = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} E_x(x,y,z,t)$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)E_x(x,y,z,t)=\frac{1}{c^2}\frac{\partial^2}{\partial t^2}E_x(x,y,z,t) \qquad [4]$$

One way, very much popular in physics, to solve said partial differential equations, is by the method of SEPARATION OF VRIABLES.

The basic strategy is: looking for a solution in the form of products of functions, of which each depends on only one of the coordinates. That is to say:

$$\boldsymbol{E}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},) = \boldsymbol{X}(\boldsymbol{x})\boldsymbol{Y}(\boldsymbol{y})\boldsymbol{Z}(\boldsymbol{z})\mathbf{T}(\mathbf{t}) \qquad [5]$$

Introducing Eq. 5 into 4 yields:

$$X''(x)Y(y)Z(z)T(t) + X(x)Y''(y)Z(z)T(t) + X(x)Y(y)Z''(z)\,T(t) = \frac{1}{c^2}X(x)Y(y)Z(z)\ddot{T}(t) \qquad [6]$$

Dividing Eq. 6 by $X(x)Y(y)Z(z)\mathrm{T}(\mathrm{t})\ yields$:

$$\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}=\frac{1}{c^2}\frac{\ddot{T}}{T} \qquad [7]$$

Since Eq. [7] holds true for every point in space (each value of x, y and z) and for every time point, each of the components of Eq. 7 must equal constant.
This is to say:

$$\frac{X''}{X}=-a^2;\ \ \frac{Y''}{Y}=-b^2\ ;\frac{Z''}{Z}=-d^2\ \text{ and }\ \ \frac{1}{c^2}\frac{\ddot{T}}{T}=\ -k^2 \qquad [8]$$

Rewriting Eq. 7 one gets:

$\frac{X''}{X} = -a^2 \Longrightarrow X''+a^2X = 0$; *and similarly* $Y''+b^2Y = 0$; *and* $Z''+d^2Z = 0$; *and* $\frac{1}{c^2}\frac{\ddot{T}}{T} = -k^2 \Longrightarrow T''+k^2c^2T = 0$

We choose the constants to be negative: $-k^2$ and not positive $+k^2$, since the latter results in a nonphysical solution:

$$-\infty < x < \infty$$

$$\frac{X''}{X} = \begin{cases} +k^2 \Longrightarrow a^2 = k^2; \quad a = k \Longrightarrow X(x) \propto e^{kx}, a\ non\ physical\ solution \\ -k^2 \Longrightarrow a^2 = -k^2; \quad a = \pm ik \Longrightarrow X(x) \propto e^{\pm ikx}, a\ physical\ solution \end{cases}$$

Therefore, the solutions of the ratio functions in [8] are to describe harmonic function, i.e. sines, cosines, and their combination. We choose the following

X=e^{iax} ; Y=e^{iby} ; Z=e^{idz} and T=e^{ikct}. Consequently,

$$\boldsymbol{E}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},) = \boldsymbol{X}(\boldsymbol{x})\boldsymbol{Y}(\boldsymbol{y})\boldsymbol{Z}(\boldsymbol{z})\mathbf{T}(\mathbf{t}) = E_0 e^{i(ax+by+dz-kct)} = = E_0 e^{i[(a_x\hat{x}+b_y\hat{y}+d_z\hat{z})\cdot(x\hat{x}+y\hat{y}+z\hat{z})-kct]} =$$

$$= \boldsymbol{E_0 e^{i(\bar{k}\cdot\bar{r}-kct)}},$$ where $k \to \bar{k} \equiv a_x\hat{x} + b_y\hat{y} + d_z\hat{z}$

or $\boldsymbol{E}(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},)$ can be $\boldsymbol{E_0 cos(\bar{k}\cdot\bar{r} - kct)}$; $\boldsymbol{E_0 sin(\bar{k}\cdot\bar{r} - kct)}$ or a combination of them, where $(\boldsymbol{\bar{k}\cdot\bar{r} - kct}) \equiv \boldsymbol{wave\ phase}$

Since the multiplicity $\{\boldsymbol{kct}\}$ must result in an angle (in radians), i.e. $\boldsymbol{kct}=\omega t \Longrightarrow k = \frac{\omega}{c} = \frac{2\pi f}{\lambda f} \Longrightarrow \boxed{k = \frac{2\pi}{\lambda}}$ and $\boxed{\omega = \frac{2\pi}{T}}$
where T is defined as the cycle (period) time.

The relation between $\bar{E}, \bar{B},\ and\ \ \bar{k}\ in\ vacuum$ (1):

Taking $\bar{E}(x,y,z,)$=$\bar{E}_0 e^{i(\bar{k}\cdot\bar{r}-\omega t)}$, then perfoming the operations $\nabla \times \overline{E}$= -$\frac{\partial \overline{B}}{\partial t}$ of Eq. 1.1.1 we get:

$$\nabla \times \overline{E} = \nabla \times \left[\bar{E}_0 e^{i(\bar{k}\cdot\bar{r}-\omega t)}\right] = curl\left[\left(E_{ox}\hat{x} + E_{oy}\hat{y} + E_{oz}\hat{z}\right)e^{i\left(k_x r_x + k_y r_y + k_z r_z - \omega t\right)}\right] =$$

$$= \hat{x}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{y}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) =$$

$$= i\left[\hat{x}\left(k_y E_z - k_z E_y\right) + \hat{y}(k_x E_x - k_x E_z) + \hat{z}\left(k_x E_y - k_y E_x\right)\right] = i\bar{k} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} = i\omega \overline{B}$$

$$\bar{k} \times \overline{E} = \omega \overline{B} \qquad [9]$$

From [9] we conclude that in electromagnetic waves in vacuum:

1. $\overline{B}$ *is perpendicular to both* $\overline{E}$ and $\bar{k}$
2. $E = \frac{\omega}{k}B = \frac{\lambda}{T}B = cB$; *i.e.* ***E*** *equal~~e~~s* ***c times B*** *in vacuum.*

The relation between $\bar{E}, \bar{B},\ and\ \bar{k}\ in\ vacuum\ (2)$:

Taking $\bar{B}(x,y,z,)$=$\bar{B}_0 e^{i(\bar{k}\cdot\bar{r}-\omega t)}$, then perfoming the operations $\nabla \times \bar{B}$= - $\mu_0 \varepsilon_0 \frac{\partial \bar{E}}{\partial t}$ of Eq. 1.1.3 we similarly get:

$$\nabla \times \overline{B} = -\mu_0 \varepsilon_0 \frac{\partial \overline{E}}{\partial t} = -\frac{1}{c^2}\frac{\partial \overline{E}}{\partial t} \longrightarrow \bar{k} \times \overline{B} = \frac{\omega}{c^2}\overline{E} = \hat{k} \times \overline{B} = \frac{\frac{\omega}{k}}{c^2}\overline{E} = \frac{1}{c}\overline{E} \qquad [10]$$

From [10] we conclude that in electromagnetic waves in vacuum:

1. $\overline{E}$ *is perpendicular to both* $\overline{B}$ and $\bar{k}$
2. $E = \frac{\omega}{k}B = \frac{\lambda}{T}B = cB$; $\quad i.e.\ \boldsymbol{E}\ equal\text{~~e~~}s\ \boldsymbol{c\ times\ B}.$
3. *Conclusion: the vectors* $\overline{E}\ \ \overline{B}\ \ and\ \ \overline{k}$ *are pendendicular to each other, i.e. forming a right hand system.*

The relation between $\bar{E}, \bar{B},\ and\ \ \bar{k}\ in\ vacuum\ (3)$:

Except for the amplitudes, are the characteristic constants $\omega,\ k, f\ the\ same\ for\ E\ and\ for\ B$? Is there phase difference between the wave fields?

Referring to $\omega_B\ and\ k_B, as\ of\ the\ field\ wave\ B\ and\ to\ \omega_E\ and\ k_E\ as\ of\ the\ field\ wave\ of\ E\ one\ get\ (from\ 9)$:

$$\bar{k}_E \times \overline{E} = \bar{k}_E \times \left[\bar{E}_0 e^{i(\bar{k}_E\cdot\bar{r}-\omega_E t)}\right] = \omega_B \bar{B}_0 e^{i(\bar{k}_B\cdot\bar{r}-\omega_B t+\delta)} \qquad [11],$$

where δ is the phase difference between the waves.

Next, dividing the left side of Eq. 11 by its right side yields **unity** (1), and assuming that $\bar{k}_E\ is\ not\ perpendicular\ to\ \overline{E}$ and there is angle θ_{E-k} , one gets:

$$\frac{sin\theta_{E-k}\cdot k_E\cdot E_0}{\omega_B B_0} e^{i[(\bar{k}_E-\bar{k}_b)\cdot\bar{r}-(\omega_E-\omega_B)t+\delta]} \equiv 1, \qquad [12], for\ \ every\ time\ t, position\ \bar{r}\ , \delta\ and\ \theta_{E-k}.$$

This can occur only when: $\bar{k}_E = \bar{k}_b = \bar{k};\quad \omega_E = \omega_B = \omega$, δ=0 and $sin\theta_{E-k} = 1,\ i.e.\theta_{E-k} = \frac{\pi}{2}$ and it turns out that:

$$\frac{sin\theta_{E-k}\cdot k_E\cdot E_0}{\omega_B B_0} = \frac{kE_0}{\omega B_0} = \frac{1}{c}\frac{E_0}{B_0} = 1$$

Biot–Savart law

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l} \times \hat{r}}{r^2} = \frac{\mu_0 \varepsilon_0}{4\pi\varepsilon_0}\frac{dq}{dt}\frac{dl\hat{l} \times \hat{r}}{r^2} = \frac{1}{c^2}\frac{dq}{dt}dt\frac{dl}{dt}\hat{l} \times \frac{\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{1}{c^2}\vec{v} \times \frac{dq}{4\pi\varepsilon_0 r^2}\hat{r} =$$

$$= \frac{1}{c^2}\vec{v} \times d\vec{E} \quad \Rightarrow \vec{E} \perp \vec{B}.$$

$$in\ case\ v \to c \ \Rightarrow\ B = \frac{E}{c}$$

Can $\bar{E}$ *or* $\bar{B}$ have a component vibrating in the direction of $\bar{k}$?

Suppose a plane front wave propagating towards $\hat{x}, i.e. \bar{E}(x, y, z,)=\bar{E}_0 e^{i(k_x r_x - \omega t)}$, Therefore, **on this plane**, where $k_x r_x$=constant, $\bar{E}$ *and* $\bar{B}$ *are* independent of x and of y, hence the derivatives $\frac{\partial}{\partial y}$ *and* $\frac{\partial}{\partial z}$ equal 0. Next, since

$$\text{div}\bar{E} = 0 = \frac{\partial E_x}{\partial x} + \underbrace{\frac{\partial E_y}{\partial y}}_{=0} + \underbrace{\frac{\partial E_z}{\partial z}}_{=0} = \frac{\partial E_x}{\partial x} \Longrightarrow \boxed{\boldsymbol{E_x = \text{constant in space.}}}$$ Similarly,

$$\text{div}\bar{B} = 0 = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial x} \Longrightarrow \boxed{\boldsymbol{B_x = \text{constant in space}}}.$$ On the other hand:

$$\nabla \times \bar{E} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} = 0 & \frac{\partial}{\partial z} = 0 \\ E_X & E_y & E_z \end{vmatrix} = \hat{x} \cdot 0 - \hat{y}\frac{\partial E_z}{\partial x} + \hat{\boldsymbol{z}}\frac{\partial E_y}{\partial x} = -\left(\dot{B}_X \hat{x} + \dot{B}_y \hat{y} + \dot{B}_z \hat{z}\right) \Longrightarrow \dot{\boldsymbol{B}}_{\boldsymbol{X}} = \boldsymbol{0} \Longrightarrow \boxed{\boldsymbol{B_X(t) = Constant.}}$$

The same treatment with $\nabla \times \bar{B} = \frac{1}{c^2}\dot{\bar{E}}$ yields: $\dot{\boldsymbol{E}}_{\boldsymbol{X}} = \boldsymbol{0} \Longrightarrow \boxed{\boldsymbol{E_X(t) = Constant.}}$

Conclusion: $\boldsymbol{E_x}$ and $\boldsymbol{B_x}$ are space and time independent and hence, even not zero, cannot be a wave. That is to say the electromagnetic waves are transverse, whereas the electric and the magnetic fields are perpendicular to each other and both, vertical to the direction of propagation ($\hat{k}$), while in phase and their real amplitudes are related by:

$$\boldsymbol{B_0 = \frac{k}{\omega} E_0 = \frac{1}{c} E_0}.$$

$$\frac{\partial \boldsymbol{P_{Q,M}}}{\partial \boldsymbol{V}} = \text{U} = -\bar{\boldsymbol{J}} \cdot \bar{\boldsymbol{E}}_{\boldsymbol{Q,M}} = \bar{\boldsymbol{H}} \cdot \frac{\partial \bar{\boldsymbol{B}}}{\partial \boldsymbol{t}} + \bar{\boldsymbol{E}} \cdot \frac{\partial \bar{\boldsymbol{D}}}{\partial \boldsymbol{t}} + \boldsymbol{div}(\bar{\boldsymbol{E}} \times \bar{\boldsymbol{H}}) \qquad \text{[e]}$$

Replacing the minus signs in [e] and Integration over space $\boldsymbol{V}$ yields:

$$P_{Q,M} = \int_{\boldsymbol{V}} (\bar{J} \cdot \bar{E}_{Q,M}) dV = -\int_{V} \left(\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} \right) dV - \int_{V} div(\bar{E} \times \bar{H}) dV \quad [f]$$

Generalization of Joule's law, representing the rate of power dissipated in the volume $\boldsymbol{V}$.

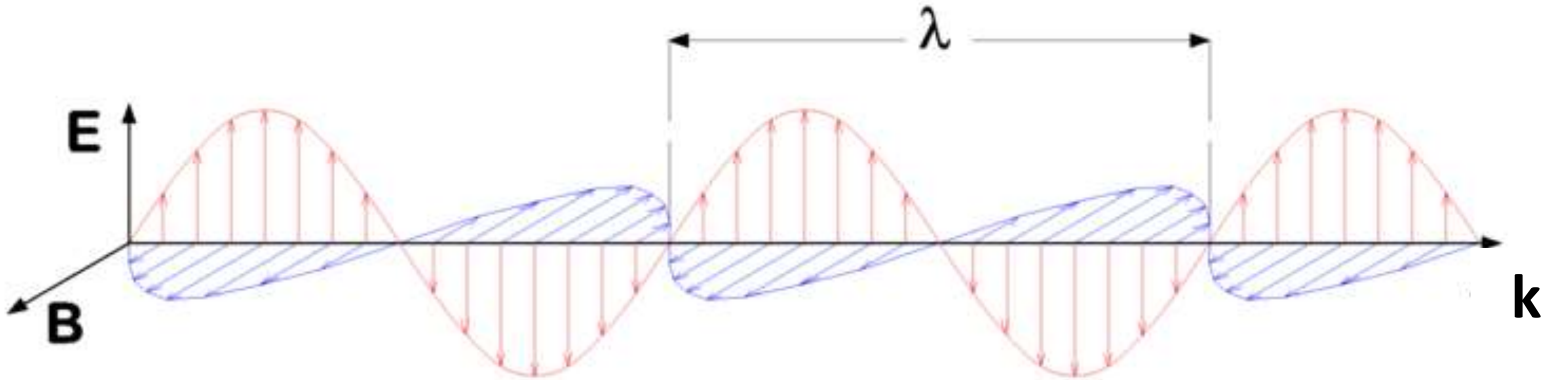
Loss (negative sign) rate of "stored" electrical and magnetic (static fields) energy within the volume $\boldsymbol{V}$. The terms in brackets are the magnetic and electric energy **densities**, u_m and u_e , respectively .

Utilizing divergence theorem this expression equals:

$$\oint_s (\bar{E} \times \bar{H}) ds$$

Conservation of energy dictates that this term must present the flow rate of energy inward/outward through the surface s enclosing the volume V. Hence, the vector $\bar{E} \times \bar{H}$ is a measure of the rate of energy flow per unit area at any point on the surface s.

The electromagnetic wave



$Remember$: $B\ is\ c\ times\ smaller\ than\ E$

Wave characteristics (1):

- λ= **the wavelength** or the special period of the wave.
- T= **the cycle time** or the temporal period of the wave.
- $c = \frac{\lambda}{T}$, **velocity** (phase velocity) **of wave propagation.**
- $f = \frac{1}{T}\{sec^{-1}\} = \frac{c}{\lambda}$, the **temporal frequency,** i.e. how many periods occurs during a unit time.
- $\omega = \frac{2\pi}{T}\{sec^{-1}\}$, **the angular temporal frequency**, i.e. number of radians per period time → $2\pi f = 2\pi\frac{c}{\lambda} = 2\pi c\tilde{\nu} = kc$
- $k = \frac{2\pi}{\lambda}$, **the wavenumber**, i.e. number of radians per unit distance. λ is the wavelength. The larger λ, the smaller k.
- $\tilde{\nu} \equiv \frac{1}{\lambda}\{cm^{-1}\}$, the **special frequency**, i.e. number of waves per unite length (how many wavelengths in a $1\ cm$?)
- $c = \frac{\lambda}{T} = \lambda f = \frac{\omega}{k}$
- $(\bar{k} \cdot \bar{r} \pm \omega t)\{rad\} = \boldsymbol{\phi}$, ***the wave phase***
- E_0 ; B_0 = wave amplitudes

Wave characteristics (2):

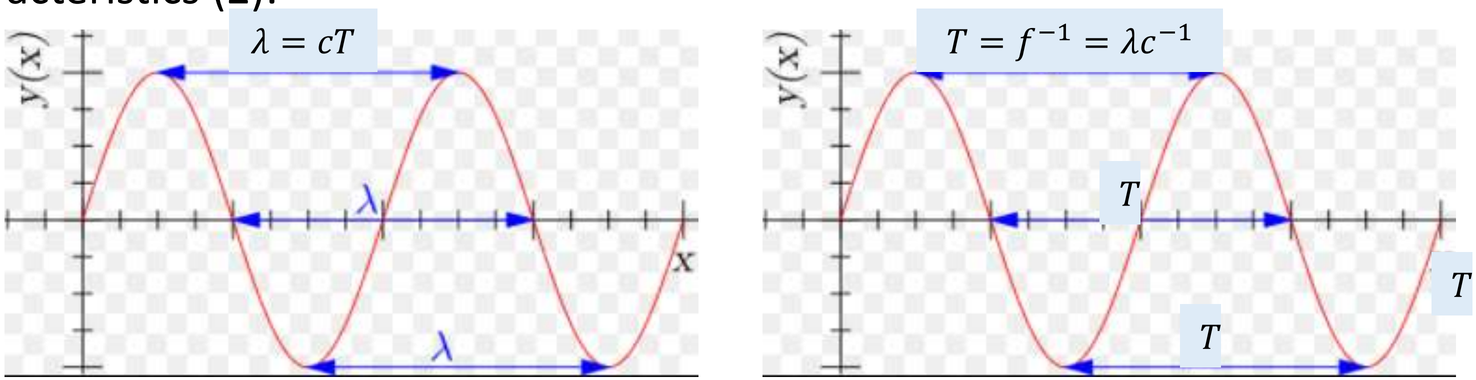


Figure 1: Wavelength λ and time period T of a wave can be measured between any two special or temporal points with the same phase, such as between crests (on top), or troughs (on bottom), or corresponding zero crossing as shown.

- $f = \frac{1}{T}\{sec^{-1}\}$; **temporal frequency**, i.e. how many times the wave reaches its maximum in a unite time (right figure).
- $\tilde{\nu} \equiv \frac{1}{\lambda}\{cm^{-1}\}$; **special frequency**, i.e. how many times the wave reaches its maximum in a unite length (left figure).
- **Longitudinal wave: the wave (medium) vibrates in the direction of its propagation.**
- **Transverse wave: the wave (medium) vibrates at right angles to the direction of its propagation.**

Biot–Savart law

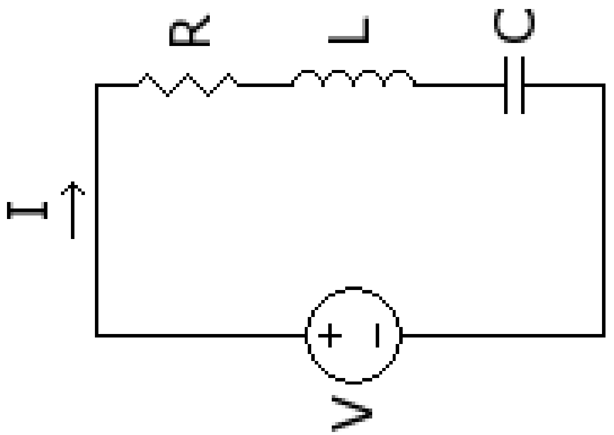
$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0\varepsilon_0}{4\pi\varepsilon_0}\frac{dq}{dt}\frac{dl\hat{l}\times\hat{r}}{r^2} = \frac{1}{c^2}\frac{dq}{dt}dt\frac{dl}{dt}\hat{l}\times\frac{\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{1}{c^2}\vec{v}\times\frac{dq}{4\pi\varepsilon_0 r^2}\hat{r} =$$

$$= \frac{1}{c^2}\vec{v}\times d\vec{E} \;\; \Rightarrow \vec{E}\perp\vec{B}.$$

$$in\ case\ v\rightarrow c \;\Rightarrow\; B = \frac{E}{c}$$

Energy and momentum in electromagnetic wave

Refreshing expressions: For convenience, we shall examine what happened in RLC electrical circuit (see figure).



$$= \boldsymbol{V_R + V_C + V_L}$$

$$V = IR + \frac{Q}{C} + L\frac{dI}{dt}$$

Multiplying the equation components by the current I one gets the equation for power (P) :

$$P_{ex} = P_{loss} + P_E + P_B$$

$$IV = I^2R + \frac{IQ}{C} + L\frac{IdI}{dt} \qquad [1]$$

Moving the expression I^2R to the left side of the equality sign of [1] the power related to the EM_P, i.e. $P_E + P_B$, is isolated, namely:

$$EM_P = IV - I^2 R = \frac{IQ}{C} + L\frac{IdI}{dt} \qquad [2]$$

Lets assume that all components ha~~s~~ the same cross section S, length l and the same volume V, $then$:

Introducing : $I = J\cdot S,\ \ J=\sigma E,\ \ Q=\sigma_q S,\ \ R=\frac{\rho l}{s}=\frac{l}{\sigma s},\ \ \mathrm{B}=\frac{\mu I}{l}\Longrightarrow I_B=\frac{Bl}{\mu},\ \ L=\frac{\phi_B}{I}=\frac{B\cdot S}{I},\ \ C=\frac{\varepsilon S}{d}\ and\ d\cdot S=\mathbf{V}\ into$:

$$IV - I^2R = \frac{IQ}{C} + L\frac{IdI}{dt} \qquad [2]$$

$$(JS)(El)\longrightarrow \boxed{\bar J\cdot\bar E\mathbf{V}}$$

$$(J^2S^2)\left(\frac{l}{\sigma s}\right)=\boxed{\frac{J^2}{\sigma}\mathbf{V}}\longrightarrow\frac{\bar J\cdot\bar J}{\sigma}\mathbf{V}$$

$$\frac{IQ}{C}=IV=\frac{\varepsilon\partial(\overset{E}{\sigma_q/\varepsilon}S)}{\partial t}Ed=\varepsilon\dot E E(Sd)=\dot D E\mathbf{V}\longrightarrow\boxed{\bar E\cdot\dot{\bar D}\mathbf{V}}$$

$$\left(\frac{BS}{I}\right)I\frac{\partial}{\partial t}\left(\frac{Bl}{\mu}\right)\longrightarrow\boxed{\bar B\cdot\dot{\bar H}\mathbf{V}}$$

Rewriting [2] with the new 'fields' expressions:

$$\bar J\cdot\bar E_{ext}\mathbf{V}-\frac{J^2}{\sigma}\mathbf{V}=\bar E\cdot\dot{\bar D}\mathbf{V}+\bar B\cdot\dot{\bar H}\mathbf{V} \qquad [3]$$

Dividing [3] by the volume $\mathbf{V}$, we get the 'The volume power density' (VPD) (i.e. the work done per unit time per unit volume) equation for electric and magnetic fields and currents:

$$\bar J\cdot\bar E_{ext}-\frac{J^2}{\sigma}=\bar E\cdot\dot{\bar D}+\bar B\cdot\dot{\bar H}\equiv\frac{\partial^2 w_{EM}}{\partial t\partial\mathbf{V}}=\cup \quad [4]$$

Input power − loss of power = ∪ , the PD of E and M fields --- all per unit volume

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Next, Remembering that $\bar{J} = \sigma \bar{E}_{Total}$, we may write:

$$\bar{J} = \sigma(\bar{E}_{ex}+\bar{E}_Q + \bar{E}_M) / \cdot \bar{J} / \sigma \quad \rightarrow \frac{J^2}{\sigma} = \bar{J} \cdot (\bar{E}_{ex}+\bar{E}_Q + \bar{E}_M) = \bar{J} \cdot \bar{E}_{ex} + \bar{J} \cdot \underbrace{(\bar{E}_Q + \bar{E}_M)}_{\equiv \bar{E}_{Q,M}} = \bar{J} \cdot \bar{E}_{ex} + \bar{J} \cdot \bar{E}_{Q,M} \Rightarrow$$

$$\Longrightarrow \underbrace{-\bar{J} \cdot \bar{E}_{Q,M}}_{loss} = \underbrace{\bar{J} \cdot \bar{E}_{ex}}_{input} - \underbrace{\frac{J^2}{\sigma}}_{heat} \qquad [a]$$

Recalling $\bar{J}$ of Maxwell's 4th equation:

$$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t} \Longrightarrow -\bar{J} = -\nabla \times \bar{H} + \frac{\partial \bar{D}}{\partial t} \qquad [b]$$

Introducing $\bar{J}$ $of\ [b]\ into\ -\bar{J} \cdot \bar{E}_{Q,M}$ of [a] yields:

$$-\bar{J} \cdot \bar{E}_{Q,M} = -\bar{E} \cdot \nabla \times \bar{H} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} \qquad [c]$$

Next, recalling from vector analysis that:

$$[26]: \quad div(\bar{E} \times \bar{H}) = \bar{H} \cdot \nabla \times \bar{E} - \bar{E} \cdot \nabla \times \bar{H} \qquad [d]$$

and substituting $-\bar{E} \cdot \nabla \times \bar{H}$ of [d] in [c], one gets:

$-\bar{J} \cdot \bar{E}_{Q,M} = -\bar{H} \cdot \nabla \times \bar{E} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} + div(\bar{E} \times \bar{H}); and\ since\ \nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} (Maxwell - Farday\ law)$, the total loss of power per unit volume $due\ to\ \bar{E}\ and \bar{B}\ is$:

$$\boldsymbol{\frac{\partial P_{Q,M}}{\partial V} = \cup = -\bar{J} \cdot \bar{E}_{Q,M} = \bar{H} \cdot \frac{\partial \bar{B}}{\partial t} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} + div(\bar{E} \times \bar{H})} \qquad [e]$$

$$P_{Q,M} = \int_{V} (\bar{J} \cdot \bar{E}_{Q,M})dV = -\int_{V} \left(\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t}\right) dV - \int_{V} div(\bar{E} \times \bar{H}\ )dV \quad [f]$$

$since \quad \bar{H} \cdot \frac{\partial \bar{B}}{\partial t} = \frac{\bar{B}}{\mu_0}\frac{\partial \bar{B}}{\partial t} = \frac{1}{2\mu_0}\frac{\partial (\bar{B})^2}{\partial t} = \frac{1}{2}\frac{\mu_0 \partial (\bar{H})^2}{\partial t}$ and $\bar{E} \cdot \frac{\partial \bar{D}}{\partial t} = \frac{1}{2}\frac{\varepsilon_0 \partial (\bar{E})^2}{\partial t}$, the first expressions in the left side of the equilibrium sign of [f] can be rewritten to yield the following equation [g]:

$$P_{Q,M} = \int_{V} (\bar{J} \cdot \bar{E}_{Q,M})dV = -\frac{\partial}{\partial t}\int_{V} \left(\frac{1}{2}\mu_0|\bar{\mathrm{H}}|^2 + \frac{1}{2}\varepsilon_0|\bar{\mathrm{E}}|^2\right) dV - \int_{V} div(\bar{E} \times \bar{H}\ )dV \quad [g]$$

though the quantities $u_m = \frac{1}{2}\mu_0|\bar{\mathrm{H}}|^2$ *and* $u_e = \frac{1}{2}\varepsilon_0|\bar{\mathrm{E}}|^2$ are known to present electric and magnetic energy densities for static fields (i.e. within a condenser or a coil). However, based on the fact that the integrands in Eq. [g] are defined at a given point, these quantities fairly represent~~s~~ the stored energy densities in the case of time-varying fields, as well. That is to say, that the correct amount of total electromagnetic energy density, u_{em}, is always obtained by assigning an amount:

$$u_{em} = u_m + u_e = \frac{1}{2}(\mu_0|\bar{\mathrm{H}}|^2 + \varepsilon_0|\bar{\mathrm{E}}|^2) = \frac{1}{2}(\bar{B} \cdot \bar{H} + \bar{D} \cdot \bar{E}) \qquad [h]$$

$recalling\ that: H = \frac{1}{\mu}B = \frac{1}{\mu c}E$, than $\mu_0|\bar{\mathrm{H}}|^2 of\ Eq. h = \mu_0 \frac{E^2}{(\mu_0 c)^2} = \rightarrow [\frac{1}{\mu_0 c^2} = \varepsilon_0] \rightarrow = \varepsilon_0 E^2$. Introducing into [h] yield:

$$\boldsymbol{u_{em}} = u_m + u_e = \frac{1}{2}(\varepsilon_0 E^2 + \varepsilon_0|\bar{\mathrm{E}}|^2) = \boldsymbol{\varepsilon_0 E^2} = \boldsymbol{\mu_0 H^2}$$

$$\frac{\partial \boldsymbol{P}_{\boldsymbol{Q},\boldsymbol{M}}}{\partial \boldsymbol{V}} = \mathrm{U} = -\bar{\boldsymbol{J}} \cdot \bar{\boldsymbol{E}}_{\boldsymbol{Q},\boldsymbol{M}} = \bar{\boldsymbol{H}} \cdot \frac{\partial \bar{\boldsymbol{B}}}{\partial \boldsymbol{t}} + \bar{\boldsymbol{E}} \cdot \frac{\partial \bar{\boldsymbol{D}}}{\partial \boldsymbol{t}} + \boldsymbol{div}(\bar{\boldsymbol{E}} \times \bar{\boldsymbol{H}}) \qquad \text{[e]}$$

Replacing the minus signs in [e] and Integration over space $\boldsymbol{V}$ yields:

$$P_{Q,M} = \int_{\boldsymbol{V}} (\bar{J} \cdot \bar{E}_{Q,M}) dV = -\int_V \left(\bar{H} \cdot \frac{\partial \bar{B}}{\partial t} + \bar{E} \cdot \frac{\partial \bar{D}}{\partial t} \right) dV - \int_V div(\bar{E} \times \bar{H}) dV \quad [f]$$

Generalization of Joule's law, representing the rate of power dissipated in the volume $\boldsymbol{V}$.

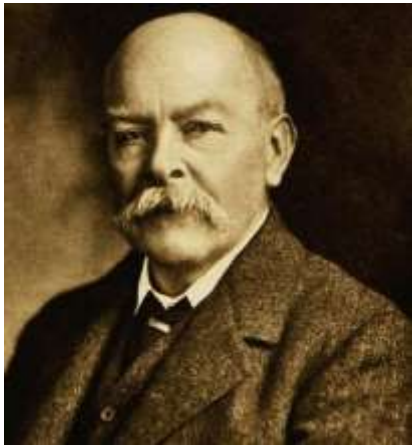
Loss (negative sign) rate of "stored" electrical and magnetic (static fields) energy within the volume $\boldsymbol{V}$. The terms in brackets are the magnetic and electric energy **densities**, u_m and u_e , respectively .

Utilizing divergence theorem this expression equals:

$$\oint_s (\bar{E} \times \bar{H}) ds$$

Conservation of energy dictates that this term must present the flow rate of energy inward/outward through the surface s enclosing the volume V. Hence, the vector $\bar{E} \times \bar{H}$ is a measure of the rate of energy flow per unit area at any point on the surface s.

$The\ vectore\ \bar{E} \times \bar{H}\ is\ the$ Poynting $vector\ \bar{N}$. It is named after its discoverer John Henry Poynting who first derived it in 1884. It represents the directional energy flux (the energy transfer per unit area per unit time) of an electromagnetic field, i.e. t$he\ power\ \boldsymbol{surface}\ density\ of\ a\ traveling\ electromagnetic\ wave$:



John Henry Poynting (1852–1914)

$$PD_{EM} = \frac{\partial^2 W_{EM}}{\partial s \partial t} = \frac{\partial P_{EM}}{\partial s} = \bar{N} \equiv \bar{E} \times \bar{H}$$

$recalling\ that: H = \frac{1}{\mu} B = \frac{1}{\mu c} E$, then:

$$|\bar{N}| = |\bar{E} \times \bar{H}| = \frac{1}{\mu c} E^{\,2} = \boldsymbol{\varepsilon_0 c \mathrm{E}^{\,2}} = \boldsymbol{\mu_0 c H^2}$$

$$\{N\} = \{\varepsilon_0 c E^{\,2}\} = C^{\,2} New^{-1} m^{-2} msec^{-1} \frac{volt^2}{m^2} = C^{\,2} New^{-1} m^{-2} msec^{-1} \frac{Juole^2/C^{\,2}}{m^2} =$$

$$= New^{-1} m^{-2} msec^{-1} \frac{New^2 m^2}{m^2} = \frac{New\ m\ sec^{-1}}{m^2} = \frac{Joule\ sec^{-1}}{m^2} = \frac{Watt}{m^2} = PD$$

The MKS unit of the Poynting **vector** is watt per square meter, $\{\bar{N}\}_{MKS} = Wm^{-2}$.

Motti: Explain the idea of vector Poynting and relate to the case of static electromagnetics.

Some magnitudes of electromagnetic waves (a):

- the electromagnetic energy per unit volume: $\frac{\partial \boldsymbol{W_{em}}}{\partial \boldsymbol{V}} = \boldsymbol{u_{em}} = \boldsymbol{\varepsilon_0 E^2} = \boldsymbol{\mu_0 H^2}$

- N, the surface power density is $\frac{\partial^2 W}{\partial t \partial s} = \frac{\partial^3 W}{\partial t \partial s}\frac{\partial l}{\partial l} = u_{em}\frac{\partial l}{\partial t} = u_{em}c \Longrightarrow \boldsymbol{N = c\, u_{em}} \quad ; \quad \boldsymbol{u_{em}} = \frac{\boldsymbol{N}}{\boldsymbol{c}} = \boldsymbol{\varepsilon_0 E^2} = \boldsymbol{\mu_0 H^2}$

- Irradiation (intensity) $\mathrm{I} \equiv \langle N \rangle_t = \langle \varepsilon_0 E^2 \rangle_t = \varepsilon_0 E_0^2 \langle cos^2\,(kr - wt) \rangle_t = \frac{1}{2}\varepsilon_0 E_0^2 \Longrightarrow \boldsymbol{I} \equiv \langle \boldsymbol{N} \rangle_{\boldsymbol{t}} = \frac{\boldsymbol{1}}{\boldsymbol{2}}\boldsymbol{\varepsilon_0 E_0^2}$

- $\wp$, the linear momentum per unit volume of electromagnetic waves:

$$\frac{d\mathfrak{p}_{volume}}{dt} = F \Longrightarrow d\mathfrak{p}_{vol} = Fdt = \frac{\partial W}{\partial x}dt = \frac{dW}{c} \Rightarrow d\wp_{unit\ vol} = \frac{d\boldsymbol{u_{em}}}{c} \Longrightarrow \wp_2 - \wp_1 = \frac{u_2}{c} - \frac{u_1}{c} \Longrightarrow$$

$$\Longrightarrow \qquad \boldsymbol{\wp} = \frac{\boldsymbol{u}}{\boldsymbol{c}} = \frac{\boldsymbol{\varepsilon_0 E^2}}{\boldsymbol{c}} = \frac{\boldsymbol{c\varepsilon_0 E^2}}{\boldsymbol{c^2}} = \frac{|\overline{\boldsymbol{E}} \times \overline{\boldsymbol{H}}|}{\boldsymbol{c^2}} \cdots\longrightarrow \frac{\boldsymbol{u}}{\boldsymbol{c}} = \frac{\boldsymbol{hf}}{\boldsymbol{c}} = \frac{\boldsymbol{h}}{\boldsymbol{c/f}} = \frac{\boldsymbol{h}}{\boldsymbol{\lambda}} = \hbar\boldsymbol{k}$$

Some magnitudes of electromagnetic waves (b):

- $\mathbb{P}$, the radiation pressure per unit volume exerted upon any surface exposed to electromagnetic radiation:

the amount of energy loss ∂W of electromagnetic wave along dx of propagation is:

$$-\frac{\partial W}{\partial x}dx = -\mathcal{F}dx = \mathbb{P}dsdx = \mathbb{P}dV \quad [i],$$

where $\boldsymbol{\mathcal{F}}$ *is an* "effevctive force" *acting along the propagation path and* $\mathbf{ds}$ *is the beam* (*wave*)*cross section.*

Consequently, from $[i]$, the radiation pressure per unit volume $\mathbb{P}$ equales:

$$\mathbb{P} = -\frac{\partial u}{\partial x}dx = -\partial u = u_1\text{-}u_2,\ where\ u = \varepsilon_0 E^2\ in\ \text{vacuum}\left[\{\varepsilon_0 E^2\} = \frac{Juole}{m^3}\right].$$

$$\{\varepsilon_0 E^{\ 2}\} = C^{\ 2}New^{-1}m^{-2}\frac{volt^2}{m^2} = C^{\ 2}New^{-1}m^{-2}\frac{Juole^2/C^{\ 2}}{m^2} = New^{-1}m^{-2}\frac{New^2m^2}{m^2} = \frac{New}{m^2} = \frac{m\cdot New}{m^3} = \frac{Juole}{m^3}$$

$Therefore, \mathbb{P},$ the radiation pressure per unit volume, is actually the consequential difference between the energy per unit volume of orthogonally incident and transmitted waves. The more the illuminated media absorbs the incident wave, i.e. $u_2 \longmapsto 0\ and\ (\partial u = u_1)$, the greater the pressure is. On the other hand, the more transparent the media, the lower the pressure is.

Practically, the radiation pressure per unit volume is evaluated in a case of fully absorbed beam, i.e. $u_2 = 0$. Then,

$$\mathbb{P} = u_1 = u = \frac{cu}{c} = c\wp = c\,\frac{\bar{E}\times\bar{H}}{c^2} \Rightarrow \qquad \bar{\mathbb{P}} = \frac{\bar{\boldsymbol{N}}}{\boldsymbol{c}}$$

Some magnitudes of electromagnetic waves (c):

In the case of fully reflection (say from a mirror, where $u_2 = -u_1$), then $\mathbb{P} = 2u = 2c\boldsymbol{\wp}$

If the angle of incidence $\theta_i \neq 0$, the cosine components of the incidence, reflected and refracted waves (beams) should be considered when calculating the relevant $\mathbb{P}(\theta_i, \theta_r)$.

Interestingly enough, the PD of the incoming Sun rays, as measured on Earth, is $\mathrm{PD} = 1.4 \cdot 10^3 Wm^{-2} = 1.4KWm^{-2}$,

hence: $$u = \frac{PD}{c} = 4.7 \cdot 10^{-6} Joule \cdot m^{-3}$$

Assuming that Earth is a perfect absorber, and taking into account the directional spread of the Sun rays, then $\mathbb{P}$, the radiation pressure per unit volume exerted upon Earth surfaces is:

$$\mathbb{P} \cong \frac{1}{3} u = 1.6 \cdot 10^{-6} New \cdot m^{-2}$$

For comparison, the atmospheric pressure is:

$$1At = 10^{+5} New \cdot m^{-2}$$