EMF waves in lossless, lossy and conducting media

- Plane Wave in **Lossless** simple Media
- Plane Wave in **good conductor**
- Plane Wave in **Lossy** simple Media

EMF In Media – Chapter 3

We restrict our discussion to LIH media, that is to say:

- Linear, i.e. linear dependencies: $\overline{D} = \varepsilon\overline{E}$ and $\overline{H} = \frac{\overline{B}}{\mu}$

- Isotropic, i.e. indifferent of the directions of $\overline{k}, \overline{E}, \overline{D}, \overline{B}, and\ \overline{H}$, and

- Homogeneous, i.e. ε and μ, representative **macroscopic** characteristics of the media, are indifferent to position (do not vary from point to point) within the media.

- Within regions in the media where there are NO free charges and/or free currents, i.e. $q = J = 0$, the Maxwell Equations differ from the vacuum analogs only in replacement of $\boldsymbol{\mu_0 \varepsilon_0}$ with $\boldsymbol{\mu\varepsilon}$, becoming:

3.1.1 $\nabla \times \overline{E} = -\frac{\partial \overline{B}}{\partial t}$

3.1.2 $\nabla \cdot \overline{E} = 0$

3.1.3 $\nabla \times \overline{B} = \boldsymbol{\mu\varepsilon} \frac{\partial \overline{E}}{\partial t}$

3.1.4 $\nabla \cdot \overline{B} = 0$

Though the mathematical observation is pretty trivial, the physical implication is astonishing:

As the wave propagates through the media, the fields polarize and magnetize all the atoms/molecules and the resulting oscillating dipoles generate their own electric fields.

These induced fields combine with the original fields in such a way as to create a single wave with the same frequency but a different speed. This is responsible for the phenomenon of **transparency**.

Performing the procedures on equations 3.1.1 and 3.1.3 as done with equations 1.1.1 and 1.1.3 in Chapter 2, results in the wave equations in LIH media:

$$\nabla^2 \overline{E} = \mu\varepsilon \frac{\partial^2 \overline{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \overline{B} = \mu\varepsilon \frac{\partial^2 \overline{B}}{\partial t^2} \qquad [4]$$

Multiplying the right side of the two equations by $\frac{\mu_0\varepsilon_0}{\mu_0\varepsilon_0}$ yields $\mu_0\varepsilon_0 \frac{\mu\varepsilon}{\mu_0\varepsilon_0} = \mu_0\,\varepsilon_0 \cdot \mu_r\,\varepsilon_r =$

$$= \frac{1}{c^2}\frac{1/v^2}{1/c^2} = \frac{1}{c^2}\frac{c^2}{v^2} = \frac{n^2}{c^2}$$

$$Hence: \qquad c^2 = \frac{1}{\mu_0\varepsilon_0}; \quad v^2 = \frac{1}{\mu\varepsilon}; \quad \boxed{\mu_r = \frac{\mu}{\mu_0}; \quad \varepsilon_r = \frac{\varepsilon}{\varepsilon_0}}; \; \mu_r\varepsilon_r = \frac{c^2}{v^2} = n^2 \Longrightarrow \boxed{\boldsymbol{n = \frac{c}{v} = \sqrt{\mu_r\varepsilon_r}}}$$

Introducing the expressions into [4]: $$\nabla^2 \overline{E} = \frac{n^2}{c^2}\frac{\partial^2 \overline{E}}{\partial t^2} \quad \text{and} \quad \nabla^2 \overline{B} = \frac{n^2}{c^2}\frac{\partial^2 \overline{B}}{\partial t^2} \qquad [5]$$

Introducing the solution $E(x,y,z,) = E_0 e^{i(\overline{k}\cdot\overline{r}-kct)}$ into [5] gives: $\left[(ik)^2 - \frac{n^2}{c^2}(i\omega)^2\right]\overline{E} = 0 \Rightarrow$

$$\Rightarrow k^2_{in\ media} = \frac{n^2}{c^2}\omega^2 \Longrightarrow (\boldsymbol{f = f_0}) \Longrightarrow \boxed{\boldsymbol{k} = n\frac{2\pi f}{c} = n\frac{2\pi}{\lambda_0} = \boldsymbol{nk_0}} \Longrightarrow \boxed{\boldsymbol{\lambda = \frac{\lambda_0}{n}}}$$

Why?

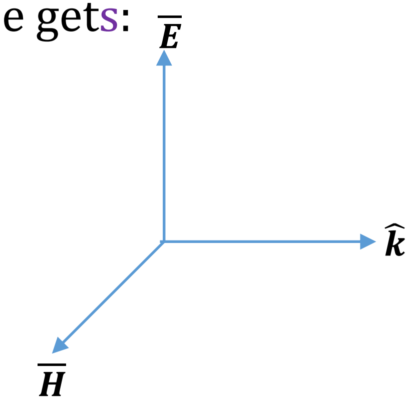
In dielectrics and biological tissues $\mu_r \cong 1$ and hence $n = \sqrt{\varepsilon_r}$

The relation between $\bar{\mathrm{E}}$ and $\bar{\mathrm{H}}$ in unbound, simple media

Recalling Maxwell's 4th equation: $\nabla \times \bar{B} = \mu\varepsilon \frac{\partial \bar{E}}{\partial t}$, and introducing into them the full expression of the fields: $\bar{E} = \bar{E}_0 e^{i(\bar{k}\cdot\bar{r}-\omega t)}$ *and* $\bar{B} = \bar{B}_0 e^{i(\bar{k}\cdot\bar{r}-\omega t)}$ we get:

$$i\bar{k} \times \bar{B} = \mu\varepsilon(-i\omega)\bar{E} \ / \frac{\times \bar{k}}{\mu} \Longrightarrow \qquad \bar{k} \times \bar{k} \times \bar{H} = -\varepsilon\omega\bar{k} \times \bar{E} \qquad [6]$$

Applying (from vector analysis): $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A}\cdot\bar{C})\bar{B} - (\bar{A}\cdot\bar{B})\bar{C}$ *onto* [6], one gets:



$$\bar{k} \times \bar{k} \times \bar{H} = \overset{=\ 0}{\left(\bar{k}\cdot\bar{H}\right)}\bar{k} - k^2\bar{H} = -\varepsilon\omega\bar{k} \times \bar{E} \quad \Longrightarrow$$

$$\Longrightarrow \bar{H} = \varepsilon\frac{\omega}{k}\hat{k} \times \bar{E} = \varepsilon v\hat{k} \times \bar{E} = \varepsilon\frac{1}{\sqrt{\mu\varepsilon}}\hat{k} \times \bar{E} = \sqrt{\frac{\varepsilon}{\mu}}\,\hat{k} \times \bar{E} = \frac{\hat{k} \times \bar{E}}{\eta}$$

$$\Longrightarrow \bar{\boldsymbol{H}} = \frac{\hat{\boldsymbol{k}} \times \bar{\boldsymbol{E}}}{\boldsymbol{\eta}} \ \textit{and similarly we can show that} \ \bar{\boldsymbol{E}} = -\boldsymbol{\eta}\hat{\boldsymbol{k}} \times \bar{\boldsymbol{H}} \qquad [7]$$

Where $\boldsymbol{\eta}=\sqrt{\boldsymbol{\mu\varepsilon}}$ *is* a quantity that has units of impedance (ohms) and is defined as the ***intrinsic impedance*** of the medium. Thus for uniform plane waves in a simple lossless medium:

the ratio of the electric and magnetic fields is η and is determined only by the material properties of the medium, i.e., η, ε.

As for the units of η and its value in vacuum:

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = \sqrt{\frac{\mu_0\varepsilon_0}{{\varepsilon_0}^2}} = \sqrt{\frac{1}{c^2{\varepsilon_0}^2}} = \frac{1}{c\varepsilon_0} = \frac{1}{3\cdot 10^8 msec^{-1} 8.85\cdot 10^{-12} C^2 New^{-1} m^{-2}} =$$

$$= \frac{1}{3\cdot 8.85\cdot 10^{-4}} \frac{\frac{New\cdot m}{C}}{\frac{C}{Sec}} = 377\frac{\frac{Joule}{C} = V(olt)}{I(Amper)} = 377\Omega \cong 120\pi\Omega$$

Analogy to Ohm's law: $I = \frac{V}{R}$; From [7] $\left|\overline{\boldsymbol{H}} = \frac{\hat{\boldsymbol{k}}\times\overline{\boldsymbol{E}}}{\boldsymbol{\eta}}\right| = \frac{\boldsymbol{E}}{\boldsymbol{\eta}} \longrightarrow where$

$\boldsymbol{H}$ *is analogous to the current* $\boldsymbol{I}$, $\boldsymbol{E}$ *to the voltage* $\boldsymbol{V}$ *and* $\boldsymbol{\eta}$ *to resistance* $\boldsymbol{R}$.

Summary: *As* ε *and* η in simple LIH media are indifferent to time and position, namely they are constant, then:

The electric and magnetic vectors of plane wave fields are perpendicular to one another. In lossless media $\boldsymbol{E}$ *and* $\boldsymbol{H}$ are **in phase** and both proportional to $\boldsymbol{p}(\boldsymbol{z} - \boldsymbol{v_p t})$, both propagate in harmony along z , reaching their maxima and minima at the same points in space and at the same times.

The orientation of $\boldsymbol{E}$ *and* $\boldsymbol{H}$ *is such that* $\boldsymbol{E}\times\boldsymbol{H}$ *is in the direction of* $+\boldsymbol{z}$, which is the direction of propagation of the wave.

Plane Wave in Lossy Media

Electromagnetic applications involve the interaction between electric and magnetic fields and matter.

The important parameters in the macroscopic levels are $\varepsilon, \mu \; and \; \sigma$.

Most media exhbit nonzero conductivity, σ, or complex permativuty, $and\ hence\ can\ absorb\ EM\ energy, resulting\ in\ attenuation$ of $EM\ wave\ while\ propogating\ thruogh\ the\ medium$.

The loss of energy is **frequency dependent** and hence determines its application range. For instance: air is quite transparent over the radio and microwave ranges, yet it is highly lossy medium at optical frequencies.

The electric field of a propagating wave within a conductive media induces the conductive currents $\boldsymbol{J}_c = \sigma \boldsymbol{E}$.

$These\ curents\ are\ in\ phase\ with\ the\ wave\ elecric\ field, and\ cause\ dissapation$ of some of the wave enery as heat $within$ $the\ material\ in\ a\ rate\ power\ per\ unite\ volume\ given\ by\ \boldsymbol{E} \cdot \boldsymbol{J}$.

Uniform Plane Wave Propagation in a good Conductor

A. Reminder: we show that for a plane wave in a simple, source free, and lossless medium ($\rho = j = \sigma = 0$) that (4^{th} Max. Eq.):

$$\nabla \times \overline{H} = \varepsilon \frac{\partial \overline{E}}{\partial t} = -i\omega \boxed{\varepsilon} \overline{E}; where\ \varepsilon\ is\ a\ real\ quantity, i.e.\ \varepsilon = \varepsilon^{Real}\ ; H = \frac{\boldsymbol{E}}{\boldsymbol{\eta}};\ \ \boldsymbol{B} = \frac{\boldsymbol{E}}{\boldsymbol{c}}$$ and

hence there is no phase difference between $\overline{H}$, $\overline{B}, \overline{E}\ and\ \overline{D}$. However,

B. In source-free conductive media, $\rho = 0,\ \ j_c \neq 0, and\ \sigma \neq 0.\ \ Then\ we\ have:$

$$\nabla \times \overline{H} = \bar{J} + \varepsilon \frac{\partial \overline{E}}{\partial t} = \sigma \overline{E} - i\omega\varepsilon \overline{E} = (\sigma - i\omega\varepsilon)\overline{E} = -i\omega \underbrace{\boxed{\left(\varepsilon^R + i\frac{\sigma}{\omega}\right)}}_{} \overline{E};$$

$$\boxed{\varepsilon_{\ eff} \longrightarrow \varepsilon_{eff}^{Im} = \frac{\sigma}{\omega}}$$

The reciprocal of the medium impedance ($\frac{1}{\eta}$) would then be:

$$\frac{1}{\eta}=\sqrt{\frac{\varepsilon_{\ eff}}{\mu}}=\frac{\left[\left(\sqrt{(\varepsilon^R)^2+(\varepsilon^I)^2}\right)^{1/2}\cdot e^{i\phi}\right]^{1/2}}{\sqrt{\mu}}=\frac{\sqrt{\varepsilon^R}\cdot\left(1+\left(\frac{\varepsilon^I}{\varepsilon^R}\right)^2\right)^{1/4}\cdot e^{\frac{i\phi}{2}}}{\sqrt{\mu}}\Longrightarrow \boldsymbol{\varepsilon^I}=\frac{\boldsymbol{\sigma}}{\boldsymbol{\omega}}=\frac{\sqrt{\varepsilon^R}\cdot\left(1+\left(\frac{\sigma}{\varepsilon^R\omega}\right)^2\right)^{1/4}\cdot e^{\frac{i\phi}{2}}}{\sqrt{\mu}}$$

In a good conductor ($\rho=0,\ j_c\neq 0, and\ \sigma\neq 0$): $\varepsilon_{eff}=\varepsilon+i\frac{\sigma}{\omega}=\varepsilon^R+i\varepsilon^I$, *out of which* the loss thangent of the medium is:

$$tan\phi_{loss}=\frac{\varepsilon^I}{\varepsilon^R}=\frac{\sigma}{\varepsilon^R\omega}$$

Note: in a good condutor and at frequencies of visibele light $\frac{\boldsymbol{\sigma}}{\boldsymbol{\omega}}\gg\mathbf{1}$ *and hence*

$$\begin{cases}\phi_{loss}\rightarrow\frac{\pi}{2}\\ \frac{1}{\eta}=\sqrt{\frac{\varepsilon^R}{\mu}}\cdot\left(\frac{\sigma}{\varepsilon^R\omega}\right)^{1/2}\cdot e^{i\frac{\pi}{4}}\end{cases}$$

And the relation between $\overline{\mathrm{E}}$ and $\overline{\mathrm{H}}$ in that case is:

$$H = \frac{E}{\eta} = \underbrace{\sqrt{\frac{\varepsilon^R}{\mu}}}_{=\eta(\sigma = J = 0)} \cdot \underbrace{\left(\frac{\sigma}{\varepsilon^R \omega}\right)}_{>>1}^{\frac{1}{2}} \cdot E e^{i\frac{\pi}{4}} = \left(\frac{\sigma}{\mu\omega}\right)^{\frac{1}{2}} \cdot E e^{i\frac{\pi}{4}} \quad \Longrightarrow$$

a. $H, B > E$ *in contrast to* $H, B < E$, *as is in a lossless medium* (lm).

b. *In lossy (conductor) medium the magnetic field lags the electric field by a phase difference which, in a good conductor, may reach* $\sim \frac{\pi}{4}$.

EMF in good conductor (Copper, Aluminum, Silver, Etc.): Approximation approach

Dielectric component $= \frac{\omega^2}{v^2}$

We showed that In exhalent conductor, $\frac{\sigma}{\varepsilon\omega}$>>1. Applying that condition into $\nabla^2\overline{E} = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2}$:

$$\Longrightarrow (ik)^2\overline{E} = [\mu\varepsilon_{eff}(-i\omega)^2]\overline{E} \Longrightarrow k \rightarrow \tilde{k} \Longrightarrow \tilde{k}^2 = \mu\varepsilon_{eff}\omega^2 = \mu\left(\varepsilon^R + i\frac{\sigma}{\omega}\right)\omega^2 = \mu\varepsilon^R\omega^2 + i\mu\sigma\omega \quad [10]$$

The ratio between the two components in the right side of Eq. 10 in the case of $\frac{\sigma}{\varepsilon^R\omega}$>>1 is:

$$\frac{\mu\varepsilon^R\omega^2}{\mu\sigma\omega} = \frac{\varepsilon^R\omega}{\sigma} \ll 1 \ and\ hence\ the\ expression\ \mu\varepsilon^R\omega^2 can\ be\ neglected\ in\ Eq.10, yielding:$$

$$\tilde{k}^2 = i\mu\sigma\omega \Longrightarrow \tilde{k} = \sqrt{\mu\sigma\omega}\cdot i^{1/2} = \sqrt{\mu\sigma\omega}\cdot\sqrt{\left(\mathrm{e}^{\frac{i\pi}{2}}\right)} = \sqrt{\mu\sigma\omega}\cdot\mathrm{e}^{\frac{i\pi}{4}} = \sqrt{\mu\sigma\omega}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{\frac{\mu\sigma\omega}{2}}(1+i) \equiv \frac{1}{\delta}(1+i) \Longrightarrow$$

$$\tilde{k} = |\tilde{k}|e^{i\phi}, where\ \tilde{k} = a + ib, but\ in\ a\ good\ conductor:$$

$$\boldsymbol{a = b = \frac{1}{\delta} = \sqrt{\frac{\mu\sigma\omega}{2}} = \sqrt{\frac{\mu\sigma 2\pi f}{2}} = \sqrt{\mu\sigma\pi f}\ and\ \phi = \frac{\pi}{4}} \qquad [\mathbf{10a}]$$

$In\ a\ good\ conductor, the\ expression\ for\ \overline{E}\ of\ a\ wave\ propagating\ along\ the\ z\ axseis$

$and\ vibreates\ along\ the\ x\ axseis, would\ be$:

$$\boldsymbol{\overline{E}(z,t)} = \overline{E}_0 e^{-bz} e^{i(az-\omega t)}\hat{x} = \boldsymbol{\overline{E}_0 e^{-\frac{z}{\delta}} e^{i\left(\frac{z}{\delta}-\omega t\right)}}\hat{x} \qquad [11]$$

Thus [11] presents an attenuated wave, i.e. the amplitude of the wave decreasing with increasing z.

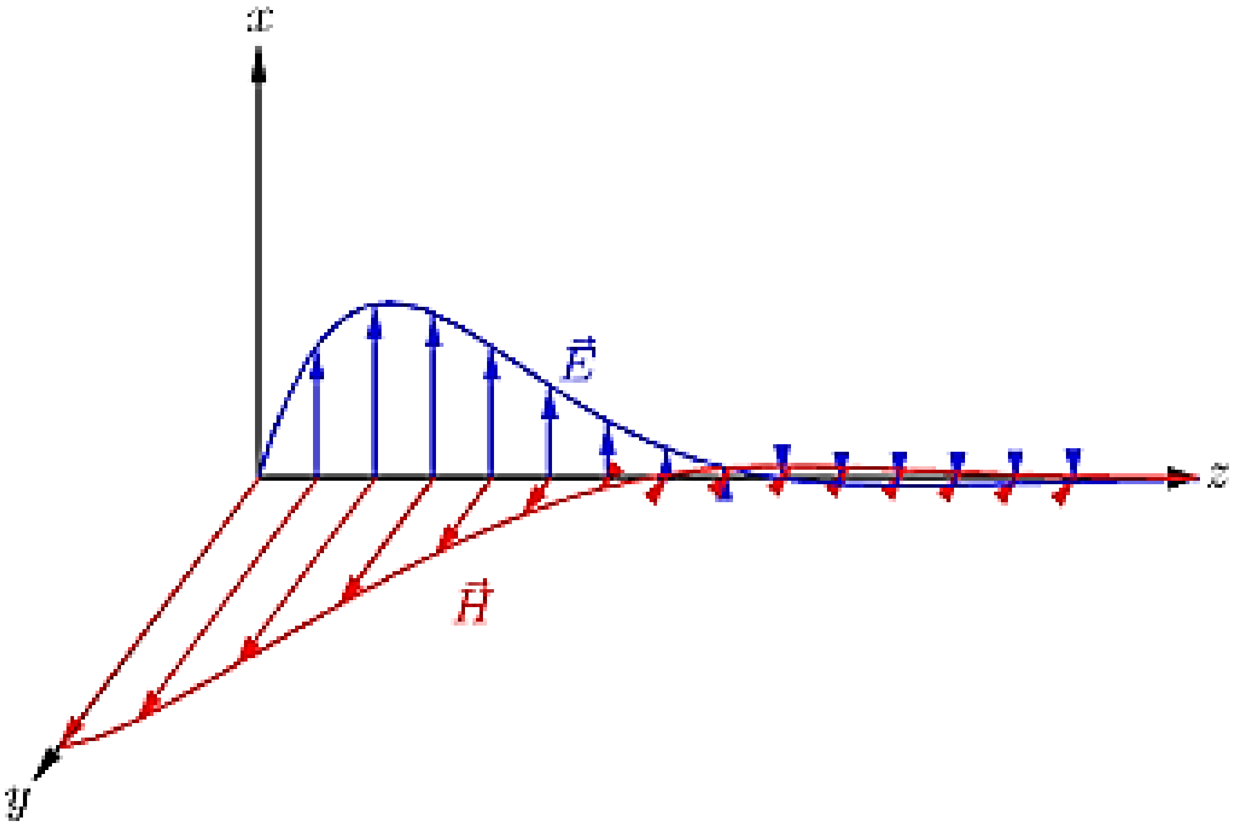
The distance it takes to reduce the amplitude by e^{-1} $is\ called\ the$

$$\boldsymbol{skin\ depth\ \delta \equiv \frac{1}{b} \rightarrow in\ our\ case = \sqrt{\frac{2}{\mu\sigma\omega}} = \sqrt{\frac{1}{\mu\sigma\pi f}}}$$

$from\ [11]\ we\ may\ comprehend\ that\ the\ practical\ wave\ \#\ in\ conductor\ is$ $\boldsymbol{a = \frac{2\pi}{\lambda_c}\ or\ \lambda_c = \frac{2\pi}{a} = 2\pi\delta = 2\sqrt{\frac{\pi}{\mu\sigma f}}}$

So, regarding good conductors, we may say that the phase velocity $\boldsymbol{v_{p.con}}$ and the wavelength $\boldsymbol{\lambda_c}$ in it are:

$$\boldsymbol{v_{p.con}} = \frac{\omega}{a} = \omega \cdot \sqrt{\frac{2}{\mu\sigma\omega}} = \boldsymbol{\sqrt{\frac{2\omega}{\mu\sigma}}} \qquad ; \qquad \boldsymbol{\lambda_c = 2\sqrt{\pi/\mu\sigma f}}$$



Electric and magnetic fields in a plane EM wave in a conductor. The wave propagate in the $+z\ direction$.

N: Poynting Vector in a good conductor

Energy per unit volume of electromagnetic wave in lossless and lossy (good conductor) media

In **lossless** media relation $\frac{u_e}{u_m}$ is:

$$\frac{u_e}{u_m}=\frac{\frac{1}{2}\bar{E}\cdot\bar{D}}{\frac{1}{2}\bar{B}\cdot\bar{H}}=\frac{\varepsilon E^2 \rightarrow \left(\varepsilon\mu=\frac{1}{v^2}\right)=\frac{E^2}{\mu v^2}}{\frac{1}{\mu}B^2 \rightarrow \left(B=\frac{E}{v}\right)=\frac{E^2}{\mu v^2}}=1$$

In **good conducting media (gcm)** we showed that $\mathrm{H}=\left(\frac{\sigma}{\mu\omega}\right)^{\frac{1}{2}} E e^{i\frac{\pi}{4}}$. Hence:

$$\frac{u_e}{u_m}=\frac{\frac{1}{2}\bar{E}\cdot\bar{D}}{\frac{1}{2}\bar{B}\cdot\bar{H}}=\frac{\varepsilon\bar{E}\cdot\bar{E}^*}{\mu\bar{H}\cdot\bar{H}^*}=\frac{\varepsilon E^2}{\mu\frac{\sigma}{\mu\omega}E^2}=\frac{\omega\varepsilon}{\sigma}\Longrightarrow \boldsymbol{\frac{u_m}{u_e}=\frac{\sigma}{\omega\varepsilon}\gg 1}$$

What is the physical mechanism which explains the result: $\boldsymbol{\frac{u_m}{u_e}\gg 1}$

Dielectric dissipative media: general treatment (The relation between $\overline{\mathrm{E}}$ and $\overline{\mathrm{H}}$ in unbuond, simple Lossy media)

Suppose we have a media with both conductance σ and imaginary permittivity $\tilde{\varepsilon} = \varepsilon^R + i\varepsilon^I$, Hence:

$$\nabla \times \overline{H} = \bar{J} + \tilde{\varepsilon}\frac{\partial \overline{E}}{\partial t} = (\sigma - i\omega\tilde{\varepsilon})\overline{E} = (\sigma - i\omega(\varepsilon^R + i\varepsilon^I))\overline{E} = -i\omega\underbrace{\left(\varepsilon^R + i\underbrace{\left(\frac{\sigma}{\omega} + \varepsilon^I\right)}_{\varepsilon^I_{eff}}\right)}_{\tilde{\varepsilon}_{eff}}\overline{E} \Longrightarrow$$

$$\Longrightarrow \begin{cases} |\tilde{\varepsilon}| = \sqrt{\varepsilon\varepsilon^*} = [(\varepsilon^R)^2 + (\dfrac{\sigma}{\omega} + \varepsilon^I)^2]^{1/2} \\[2ex] tan\phi_{loss} = \dfrac{\dfrac{\sigma}{\omega} + \varepsilon^I}{\varepsilon^R} \end{cases}$$

Therefore again: $\tilde{\varepsilon} = |\tilde{\varepsilon}|e^{i\phi}$ and consequently

$$H = \frac{1}{\eta}E = \sqrt{\frac{|\tilde{\varepsilon}|}{\mu}}\, e^{i\phi/2}E$$

Obviously, for this medium ϕ may differ from $\frac{\pi}{2}$

Next, the influence of the dissipative media upon the traveling electromagnetic wave media, is realized through the impact of its complex permittivity $\tilde{\varepsilon}$ on the wave # $\bar{k}$ (*the dispersion ratio*) as follows:

We recall in vacuum and lossless media that: $\nabla^2\overline{E} = (ik)^2\overline{E} = \mu\varepsilon\frac{\partial^2\overline{E}}{\partial t^2} = \mu\varepsilon(-i\omega)^2\overline{E} \Longrightarrow k^2 = \mu\varepsilon\omega^2 = \left(\frac{w}{v}\right)^2 \longrightarrow k = \frac{w}{v}$

However, in lossy media $\varepsilon_{eff} = \varepsilon^R + i\varepsilon^I \Longrightarrow k^2 = \mu(\varepsilon^R + i\varepsilon^I)\omega^2 = \underbrace{\mu\varepsilon^R\omega^2}_{\text{lossless Component of } \tilde{k}^2} + \underbrace{\mu i\varepsilon^I\omega^2}_{\text{lossy Component of } \tilde{k}^2} \equiv (\tilde{\boldsymbol{k}})^2$

Out of which:

$$\tilde{k}^2 = \mu\varepsilon^R\omega^2\left(1+i\frac{\varepsilon^I}{\varepsilon^R}\right) \Longrightarrow \tilde{k} = \sqrt{\mu\varepsilon^R\omega^2}\left(1+i\frac{\varepsilon^I}{\varepsilon^R}\right)^{\frac{1}{2}} = \boldsymbol{\mathscr{k}}(1+i\boldsymbol{\mathscr{p}})^{\frac{1}{2}} = \boldsymbol{a}+\boldsymbol{ib},\ where{:}\ \mathscr{k} \equiv \sqrt{\mu\varepsilon^R\omega^2}\ \ and\ \mathscr{p} \equiv \frac{\varepsilon^I}{\varepsilon^R}.\ \ Next{:}$$

1. $\tilde{k}\tilde{k}^* = a^2+b^2 = \mathscr{k}^2(1+\mathscr{p}^2)^{1/2}$
2. $k^2 = a^2-b^2+2iab = \mathscr{k}^2(1+i\mathscr{p}) \Longrightarrow a^2-b^2=\mathscr{k}^2$ *and* $2ab = \mathscr{k}^2\mathscr{p}$

$$From\ (1+2) \Longrightarrow 2a^2 + i\cancel{\mathscr{k}^2}\mathscr{p} = \mathscr{k}^2(1+\mathscr{p}^2)^{\frac{1}{2}} + \mathscr{k}^2(1+i\mathscr{p}) = \mathscr{k}^2(1+\mathscr{p}^2)^{\frac{1}{2}} + \mathscr{k}^2 + i\cancel{\mathscr{k}^2}\mathscr{p}. = \mathscr{k}^2\left[(1+\mathscr{p}^2)^{\frac{1}{2}}+1\right]$$

$$a = \mathscr{k}\frac{\left[(1+\mathscr{p}^2)^{\frac{1}{2}}+1\right]^{1/2}}{\sqrt{2}} = \omega\sqrt{\mu\varepsilon^R}\frac{\left[(1+(\varepsilon^I/\varepsilon^R)^2)^{\frac{1}{2}}+1\right]^{1/2}}{\sqrt{2}};\ \ and\ from\ (1-2)$$

$$b = \mathscr{k}\frac{\left[(1+\mathscr{p}^2)^{\frac{1}{2}}-1\right]^{1/2}}{\sqrt{2}} = \omega\sqrt{\mu\varepsilon^R}\frac{\left[(1+(\varepsilon^I/\varepsilon^R)^2)^{\frac{1}{2}}-1\right]^{1/2}}{\sqrt{2}}$$

$$\Longrightarrow tan\phi_{loss} = \frac{b}{a}$$

Then , for wave propagating in Z direction (**this time the wave # $\boldsymbol{k}$ is imaginary**) :

$$\bar{E} = \bar{E}_0 e^{i(\tilde{k}z-\omega t)} = \bar{\boldsymbol{E}}_{\mathbf{0}} e^{i((a+ib)z-\omega t)} = \bar{\boldsymbol{E}}_{\mathbf{0}} \boldsymbol{e}^{-\boldsymbol{bz}} e^{-i(az-\omega t)} \; [8]$$

The right side expression is that of an attenuated wave, i.e. the amplitude of the wave decrease with increasing z.

The distance it takes to reduce the amplitude by e^{-1} *is called the*

$$\boldsymbol{skin\ (penetration)\ depth\ \ell \equiv \frac{1}{b}}.$$

$\boldsymbol{\ell}$ is a measure of how far the wave penetrates into the conductor (d~~e~~issipative media). Meanwhile, the real part of k, $i.e.$ $\boldsymbol{a}$, determines the wavelength, the propagation speed and the refraction index, in the media in the usual way:

$$\lambda = \frac{2\pi}{a}, \qquad v = \frac{\omega}{a}, \qquad n = \frac{c}{v} = \frac{ca}{\omega}$$

Like Equation [8] it is trivial to show that the same procedure works for $\bar{B}$ as well:

$$\bar{B} = \bar{B}_0 e^{i(\tilde{k}z-\omega t)} = \frac{|\tilde{\boldsymbol{k}}|}{\boldsymbol{\omega}} \bar{\boldsymbol{E}}_{\mathbf{0}} e^{i((a+ib)z-\omega t)} = \frac{|\tilde{\boldsymbol{k}}|}{\boldsymbol{\omega}} \bar{\boldsymbol{E}}_{\mathbf{0}} \boldsymbol{e}^{-\boldsymbol{bz}} e^{-i(az-\omega t)} \; [9]$$

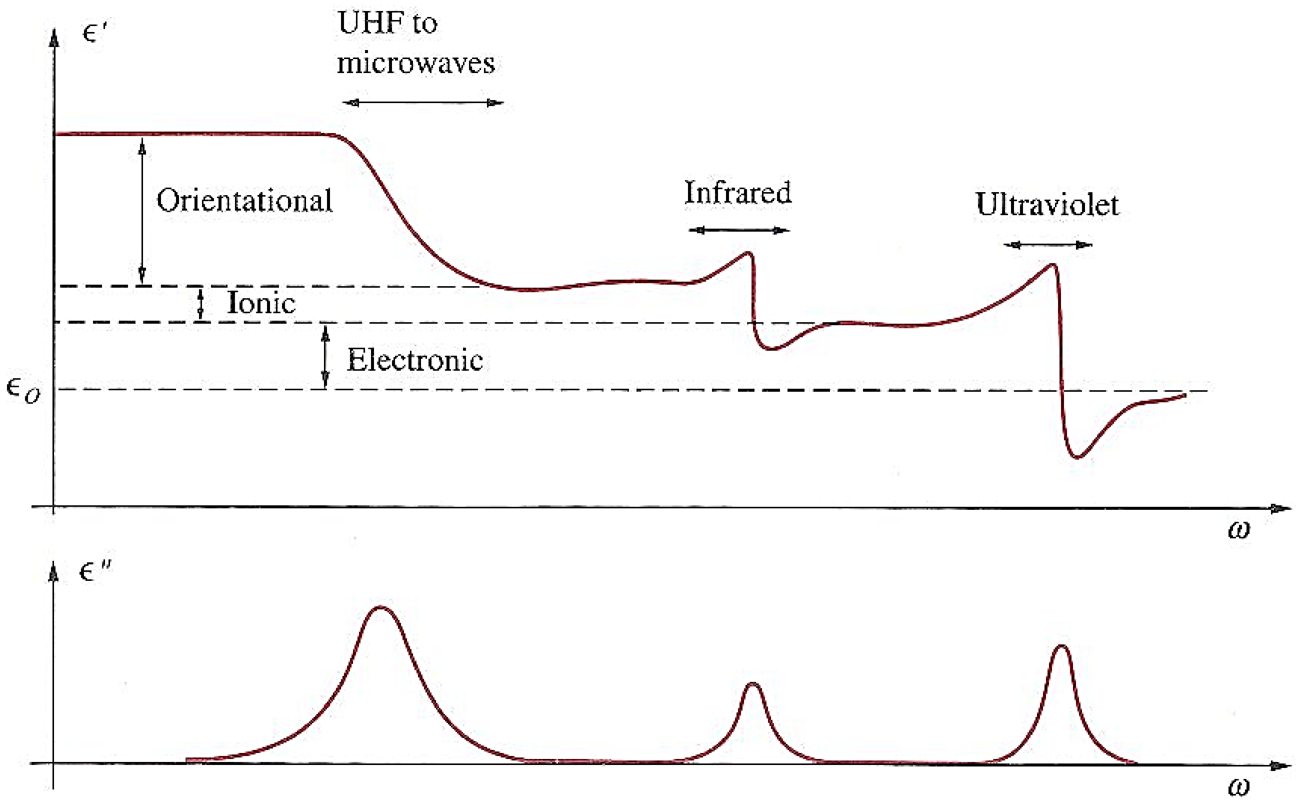
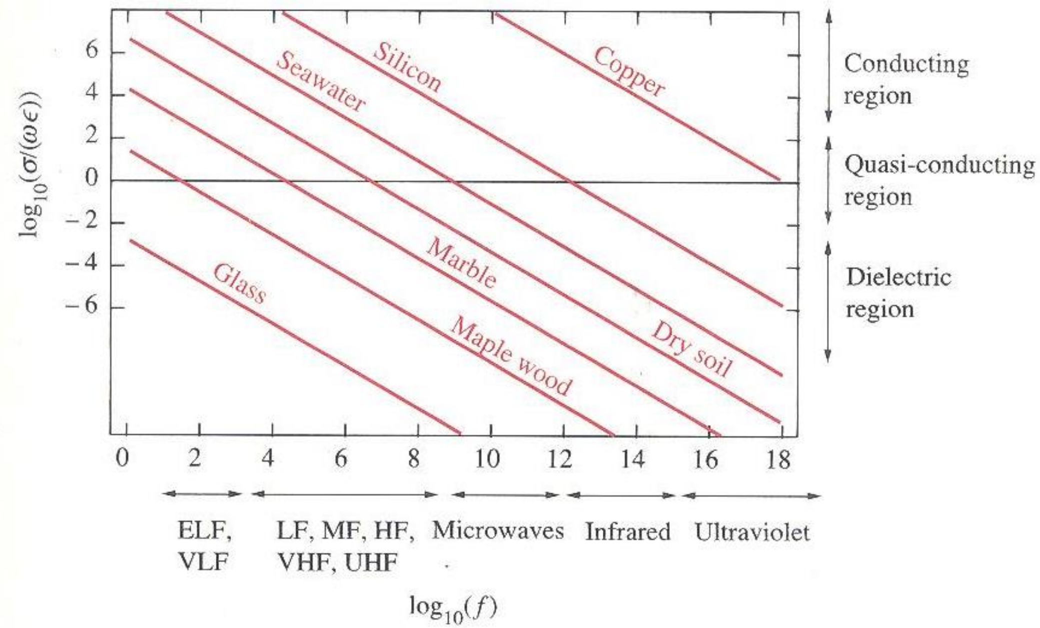


FIGURE 7.22. Dielectric constant as a function of frequency. At low frequencies, the permittivity differs from ϵ_0 by a constant multiplier. In the vicinity of the resonance, ϵ'' goes through a pronounced peak, while ϵ' generally decreases to a new level.

Loss tangent ($tan\delta_c = \sigma/\omega\epsilon$) versus frequency



For typical good conductors, both σ and ϵ are nearly independent of frequencies below the optical range.

Relative permittivity and conductivity used in the above Figure.

Medium	(dimensionless)	(S-m^{-1})
Copper	1	5.8×10^7
Seawater	81	4
Doped silicon	12	10^3
Marble	8	10^{-5}
Maple wood	2.1	3.3×10^{-9}
Dry soil	3.4	10^{-4} to 10^{-2}
Fresh water	81	$\sim 10^{-2}$
Mica	6	10^{-15}
Flint glass	10	10^{-12}

TABLE 8.3. Dielectric properties of selected materials

Material	f (GHz)	ϵ_r'	ϵ_r''	T (°C)
Aluminum oxide (Al_2O_3)	3.0	8.79	8.79×10^{-3}	25
Barium titanate ($BaTiO_3$)	3.0	600	180	26
Bread	2.45	4.6	1.20	
Bread dough	2.45	22.0	9.00	
Butter (salted)	2.45	4.6	0.60	20
Cheddar cheese	2.45	16.0	8.7	20
Concrete (dry)	2.45	4.5	0.05	25
Concrete (wet)	2.45	14.5	1.73	25
Corn (8% moisture)	2.45	2.2	0.2	24
Corn oil	2.45	2.5	0.14	25
Distilled water	2.45	78	12.5	20
Dry sandy soil	3.0	2.55	1.58×10^{-2}	25
Egg white	3.0	35.0	17.5	25
Frozen beef	2.45	4.4	0.528	−20
Honey (100% pure)	2.45	10.0	3.9	25
Ice (pure distilled)	3.0	3.2	2.88×10^{-3}	−12
Milk	3.0	51.0	30.1	20
Most plastics	2.45	2 to 4.5	0.002 to 0.09	20
Papers	2.45	2 to 3	0.1 to 0.3	20
Potato (78.9% moisture)	3.0	81.0	30.8	25
Polyethylene	3.0	2.26	7.01×10^{-4}	25
Polystyrene	3.0	2.55	8.42×10^{-4}	25
Polytetrafluoroethylene (Teflon)	3.0	2.1	3.15×10^{-4}	22
Raw beef	2.45	52.4	17.3	25
Snow (fresh fallen)	3.0	1.20	3.48×10^{-4}	−20
Snow (hard packed)	3.0	1.50	1.35×10^{-3}	−6
Some glasses (Pyrex)	2.45	~4.0	0.004 to 0.02	20
Smoked bacon	3.0	2.50	0.125	25
Soybean oil	3.0	2.51	0.151	25
Steak	3.0	40.0	12.0	25
White onion (78.7% moisture)	2.45	53.8	13.5	22
White rice (16% moisture)	2.45	3.8	0.8	24
Wood	2.45	1.2 to 5	0.01 to 0.5	25
Water	24			

Examples:

Microwave heating of milk: the dielectric properties of milk with 7.3% moisture content at $20^{\circ}C\ and\ 3GHz$ are $\epsilon_r' = 51\ and\ tan\ \delta_c = 0.59$. Calculate (a) ϵ_r'' and (b) the average dissipated power per unit volume if the peak electric field inside the dielectric is $30kVm^{-1}$.

Solution:

(a) From $tan\ \delta_c = 0.59 = \epsilon_r''/\epsilon_r' = \epsilon_r''/51 \implies \epsilon_r'' = 51 \cdot 0.59 \cong 30.1$

Recall that $u_e^{loss} = \bar{J}_c \cdot \bar{E} = \sigma\bar{E} \cdot \bar{E} = \sigma E^2$. However, when $\omega\epsilon'' is$ high an alternating current density $\bar{J}_D = \omega\epsilon''\bar{E}$ flows within the dielectric, yielding an instantaneous loss $u_e^{loss} = \bar{J}_D \cdot \bar{E} = \omega\epsilon''\bar{E} \cdot \bar{E} = \omega\epsilon'' E^2.$

Consequently, the average power dissipated per unite volume for time harmonic electric field $E = E_{peak} \cdot \cos(\omega t)$ would be $\langle u_e^{loss} \rangle = \frac{1}{2}\omega\epsilon'' E_{peak}^2$. Hence we have:

(b) $\frac{1}{2}\omega\epsilon'' E_{peak}^2 \cong \frac{1}{2}(2\pi \times 3 \times 10^9 rad/\sec)(30.1 \times 8.85 \times 10^{-12} F/m)(30kV/m)^2 \cong$

$$\cong 2.26 \times \frac{10^9 W}{m^3} = 2.26W/m^3$$

$$\bar{E} = \boldsymbol{\bar{E}_0 e^{-bz}} e^{-i(az-\omega t)} \qquad [8]$$

$$\bar{B} = \frac{|\tilde{\boldsymbol{k}}|}{\boldsymbol{\omega}} \boldsymbol{\bar{E}_0 e^{-bz}} e^{-i(az-\omega t)} \qquad [9]$$

The attenuated plane waves [8, 9] satisfy the modified equations:

For any $\bar{E}_0$ and $\bar{B}_0$.

3.1.1 $\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$ 3.1.2 $\nabla \cdot \bar{E} = 0$

3.1.3 $\nabla \times \bar{B} = \mu\varepsilon \frac{\partial \bar{E}}{\partial t} + \mu\sigma\bar{E}$ 3.1.4 $\nabla \cdot \bar{B} = 0$

Yet these Maxwell~~'s~~ equations impose further constraints, which serve to determine the relative amplitudes, phase and polarization of $\bar{E}$ *and* $\bar{B}$. Obviously, $\nabla \cdot \bar{E} = \nabla \cdot \bar{B} = 0$, as shown previously, rule out any z components, i.e. the fields are ***transverse***.

For instance, lets choose $\bar{E}$ to be polarized (vibrate) along the *x* direction:

$\bar{E}(z,t) = \bar{E}_0 e^{-bz} e^{-i(az-\omega t)} \hat{x}$, then applying 3.1.3 above yields: $\bar{B}(z,t) = \bar{B}_0 e^{-bz} e^{-i(az-\omega t)} \hat{y}$

So once again, $\bar{E} \perp \bar{B}$. On the other hand, $\tilde{k}$, as any complex number can be expressed in terms of its modulus and phase:

$$\tilde{k} = |\tilde{k}| e^{i\phi}, where |\tilde{k}| = \sqrt{a^2 + b^2} = \sqrt{\mu\varepsilon^R\omega^2}\left(1 + \left(\frac{\varepsilon^I}{\varepsilon^R}\right)^2\right)^{\frac{1}{2}} \mapsto conductor \mapsto = \omega\sqrt{\mu\varepsilon^R}\left(1 + \left(\frac{\sigma}{\varepsilon\omega}\right)^2\right)^{\frac{1}{2}}$$

Summary: According to Equations [8, 9] the complex amplitudes $\tilde{E}_0 = E_0 e^{i\delta_E}$ *and* $\tilde{B}_0 = B_0 e^{i\delta_B}$ are no longer in phase; and they are related by:

$$B_0 e^{i\varphi_B} = \frac{|\tilde{k}|e^{i\phi}}{\omega} E_0 e^{i\varphi_E} \;\; ; \quad where \;\; \varphi_B - \varphi_E = \phi$$

The magnetic field *lags behind* the electric field. The (real) amplitudes of $\bar{E}$ and $\bar{B}$ are related by:

$$\frac{E_0}{B_0} = \frac{|\tilde{k}|}{\omega} = \sqrt{\varepsilon\mu\sqrt{1+\left(\frac{\sigma}{\varepsilon\omega}\right)^2}}$$

Hence, the (real) magnetic and electric fields are:

$$\overline{\boldsymbol{E}}(\boldsymbol{z},\boldsymbol{t}) = \overline{\boldsymbol{E}}_{\boldsymbol{0}} \boldsymbol{e}^{-\frac{\boldsymbol{z}}{\boldsymbol{\delta}}} \boldsymbol{e}^{\boldsymbol{i}\left(\frac{\boldsymbol{z}}{\boldsymbol{\delta}}-\boldsymbol{\omega t}+\boldsymbol{\varphi}_{\boldsymbol{E}}\right)} \hat{x}$$

$$\overline{\boldsymbol{B}}(\boldsymbol{z},\boldsymbol{t}) = \overline{\boldsymbol{B}}_{\boldsymbol{0}} \boldsymbol{e}^{-\frac{\boldsymbol{z}}{\boldsymbol{\delta}}} \boldsymbol{e}^{\boldsymbol{i}\left(\frac{\boldsymbol{z}}{\boldsymbol{\delta}}-\boldsymbol{\omega t}+\boldsymbol{\varphi}_{\boldsymbol{E}}+\boldsymbol{\phi}\right)} \hat{y}$$

Since both $\boldsymbol{a}\ and\ \boldsymbol{b}$ are proportional to $\boldsymbol{\sigma}^{\mathbf{1/2}}$ and $\boldsymbol{\sigma}$ is large, it appears that uniform plane waves not only are attenuated heavily but also undergo a significant phase shift per unite length as they propagate in a good conductor.

On the other hand, since the phase velocity $\boldsymbol{v_{p.con}}$ and the wavelength $\boldsymbol{\lambda_c}$ are both proportional to $\boldsymbol{\sigma}^{-\mathbf{1/2}}$, they are both significantly smaller than the corresponding values in free space, i.e. $\boldsymbol{v_{p.vac}}\ and\ \boldsymbol{\lambda_{vac}}$.

For instance, for copper ($\sigma = 5.8 \cdot 10^7 Sm^{-1}$), at $\mathbf{300}\ \boldsymbol{MHz}, we\ get$:

$$\boldsymbol{v_{p.copp}} \cong \mathbf{7192}\boldsymbol{msec^{-1}}\ \ [v_{vac} = c = 3 \cdot 10^8 msec^{-1}]\ \text{and}\ \boldsymbol{\lambda_{copp}} = \mathbf{0.024}\boldsymbol{mm}\ [\lambda_{vac} = 1\ m\ ]$$

For $\mathbf{60}\ \boldsymbol{Hz}$ instance, for copper, the values are more dramatic:

$$\boldsymbol{v_{p.copp}} \cong \mathbf{3.22}\boldsymbol{msec^{-1}}\ \ [v_{vac} = c = 3 \cdot 10^8 msec^{-1}]\ \text{and}\ \boldsymbol{\lambda_{copp}} = \mathbf{53.6}\boldsymbol{m}\ [\lambda_{vac} \sim 5000 km\ ]$$

As an example of a **nonmetallic conductor, for seawater** ($\varepsilon_r = 81, \sigma = 4Sm^{-1}$), at $\mathbf{10}\ \boldsymbol{kHz}$ one gets:

$$\boldsymbol{v_{p.SW}} \cong \mathbf{1.58 \cdot 10^5}\boldsymbol{msec^{-1}}\quad \text{and}\ \boldsymbol{\lambda_{SW}} \cong \mathbf{15.8}\boldsymbol{m}\ [\lambda_{vac} \sim 30km\ ]$$

Dispersion:

The frequency dependence of the permittivity
(and refractive index)

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Maxwell's Theory treats matter as continuous, represented by the constants ε *and* μ with resulting n, unrealistically, independent of the frequency of the EMF. To understand of the dependency $n(f), i.e. dispersion,$ the atomic/molecular aspect of matter must be considered.

When a dielectric is subjected to an electric field the atomic/molecular charge distribution within it is distorted, inducing of electric dipole moments, which in turn modifies the total internal electric field.

There are *permanent polar molecules* (Figure 3.34):

Thermal agitation keeps the dipoles randomly oriented, yielding overall zero polarization.

Introduction of electric field cause dipoles alignment, i.e. causing orientational polarization

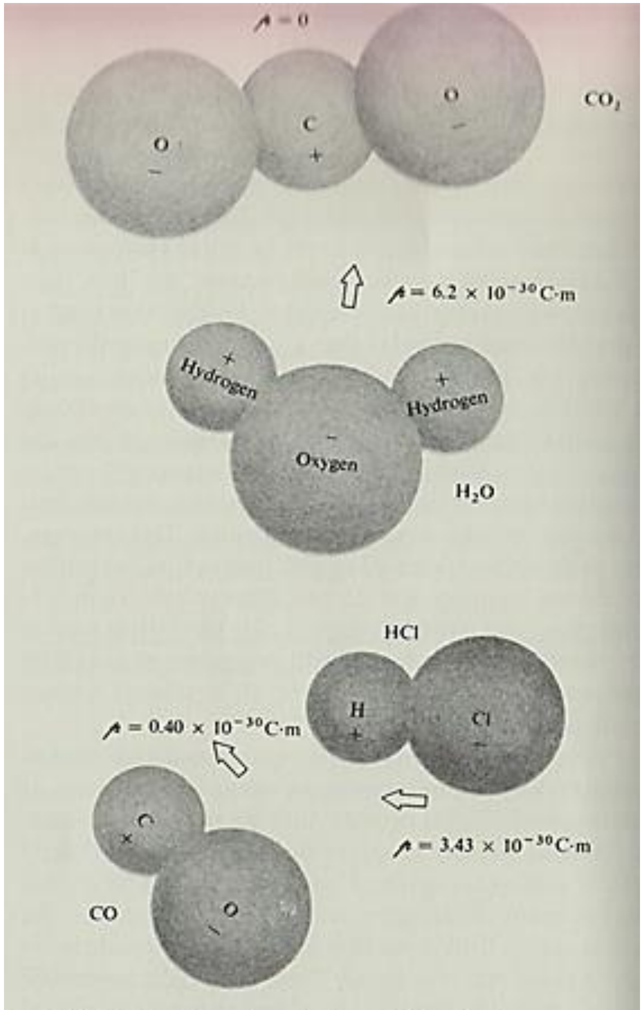


FIGURE 3.34 Assorted molecules and their dipole moments.

And in many cases *ionic/crystal polarization*

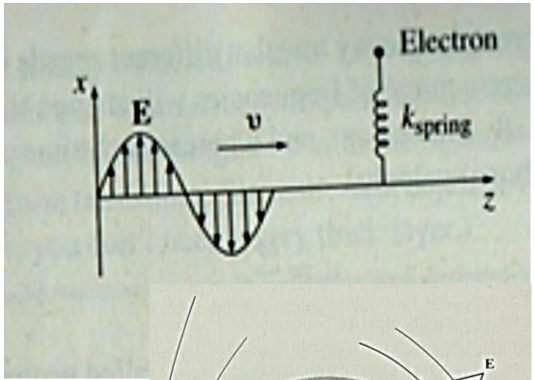
Polarization due to EMF

When a dielectric experiences EMF, the charges of its atoms/molecules is subjected to time varying forces/torques, proportional to the electric field $\overline{\boldsymbol{E}}(\boldsymbol{t})$ of the wave. Due to molecule inertia (and bounds), the higher the f is, the lesser the molecule response will be (aligning with $\overline{\boldsymbol{E}}(\boldsymbol{t})$) and consequently ε will markedly decrease.

For instance, for water $\varepsilon_r = 81 \text{ up to } 10^{10} Hz$, after which it drops quite rapidly.

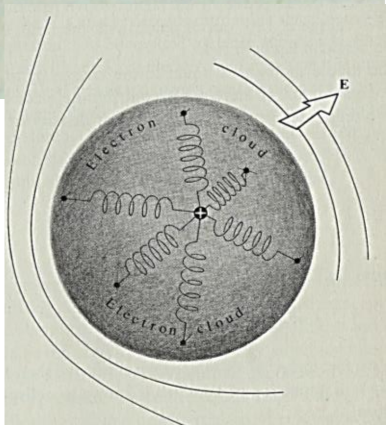
In contrast, due to their little inertia, electrons are relatively highly responsive to $\overline{\boldsymbol{E}}(\boldsymbol{t})$ even at optical frequencies ($\sim 10^{14} Hz$). $This\ makes$ the permittivity and the refraction index frequency-dependent, i.e. :

$$\varepsilon \ \rightarrow \ \varepsilon(\omega) \text{ and } n \ \rightarrow n(\omega)$$



For the sake of simplicity, within the frame of a reasonable approximation and small displacement x, we relate to the attracting electrical force acting between the positive nucleus and the electrons within an atom/molecule, as being spring-like i.e. picture an electron as attached to the end of an imaginary spring, with force constant k_{spring}:

$$F_{restoring} = -k_{spring} \cdot x = -m\omega_0^2 x$$



Where $\omega_0 = \sqrt{\frac{k}{m}}$ is the bound electron **natural or resonant** oscillation frequency.

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So we have: (1) $F_{restoring} = -k_{spring} \cdot x = -m\omega_0^2 x$ and

In that simple model the damping force will be velocity ($\dot{x}$) dependent, namely
(2) $F_{damping} = -m\gamma\dot{x}$ (the actual cause for damping is radiation by the oscillating electron. We will learn this).

The deriving EMF is the harmonic: $\overline{\boldsymbol{E}}(\boldsymbol{t}) = \overline{\boldsymbol{E}}_{\boldsymbol{0}} e^{i(kx-\omega t)} = \overline{\boldsymbol{E}}_{\boldsymbol{0}} e^{i(2\pi\frac{x}{\lambda}-\omega t)}$

However, since $\lambda_{EMF} \gg atom\ size, i.e. \Delta x$, the special wave component $2\pi\frac{x}{\lambda}$ can be neglected, leaving the EMF to be just time dependent: $\overline{\boldsymbol{E}} = \overline{\boldsymbol{E}}(\boldsymbol{t}) = \overline{\boldsymbol{E}}_{\boldsymbol{0}} e^{-i(\omega t)}$and hence the driving force to be:

$$(3) \qquad \overline{\boldsymbol{F}}_{\boldsymbol{driving}} = e\overline{\boldsymbol{E}}(\boldsymbol{t}) = e\overline{\boldsymbol{E}}_{\boldsymbol{0}} e^{i\omega t}$$

Where $\boldsymbol{e}$ is the electron charge and $\overline{\boldsymbol{E}}_{\boldsymbol{0}}$ the is amplitude of the EM wave. Putting all this into Newton's second law yields:

$m\dfrac{d^2x}{dt} = F_{restoring} + F_{damping} + F_{driving},$ $\quad$ $and\ idntroducing\ 1,2,3\ and\ dividing\ by\ the\ mass\ \boldsymbol{m}\ we\ get$:

$$\boldsymbol{\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{e}{m} E_0 e^{i\omega t}} \qquad [20]$$

Where $\boldsymbol{m}$ is the electron **mass**, and $\boldsymbol{\gamma}$ is the **damping coefficient** (the coefficient 2 was chosen for convenience). To satisfy Eq. 20, $\boldsymbol{x}$ must be a function whose second derivative is similar to itself. Furthermore, it is anticipated that the electron will follow the deriving force and hence have the same frequency. So we guess that:

$x(t) = Ce^{i\omega t}$ **[21]** and introducing **[21]** into **[20]** gives:

$(-\omega^2 + 2\gamma(i\omega) + \omega_0^2)Ce^{i\omega t} = \frac{e}{m}Ee^{i\omega t}$ out of which we get that:

$$C = \frac{eE/m}{\omega_0^2 - \omega^2 + 2i\gamma\omega}$$

multiplying both the numerator and denominator by the latter yields the complex number $\tilde{C}$:

$\tilde{C} \equiv C^R + iC^I = \frac{eE_o/m(\omega_0^2 - \omega^2 - 2i\gamma\omega)}{(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2}$ and hence [22]

$$|\tilde{C}| = \sqrt{\tilde{C}\tilde{C}^*} =$$

$$= \left\{\frac{(eE_o/m)^2\ (\omega_0^2 - \omega^2 - 2i\gamma\omega)(\omega_0^2 - \omega^2 + 2i\gamma\omega)}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^2}\right\}^{\frac{1}{2}} = \frac{eE_o/m}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} = |\tilde{C}|e^{\pm i\varphi}$$

Where: $tan\varphi = \frac{C^I}{C^R} = \frac{-2\gamma\omega}{\omega_0^2 - \omega^2}$

Introducing $\left|\tilde{C}\right|e^{\pm i\varphi}$ into [21], i.e. into $x(t) = Ce^{i\omega t}$ one gets [22]:

$$x(t) = \left|\tilde{C}\right|e^{\pm i\varphi}e^{i\omega t} = \left|\tilde{C}\right|e^{i(\omega t \pm \varphi)} = \frac{eE_o/m_e}{\left[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2\right]^{1/2}}e^{i(\omega t \pm \varphi)} \quad [22]$$

When $\frac{\omega_0^2}{\omega^2} \ll 1$, elastic scattering occurs, i.e. no energy loss (from the EMF within the substance) happened.

Since $\omega_0^2 - \omega^2 >> \gamma\omega,\ the\ damping\ element\ can\ be\ neglected.$ The closer ω is to ω_o the larger $x(t)$ is, and absorption occurs.

Polarizability (α), Electrical Susceptibility (χ_e) and the refractive index (n) in LIH medium

The ***atomic electrical dipole moment*** p is the product of its positive charge times the distance (displacement) x between the positive and negative charges. As said, when the dipole is subjected to varying EMF, both the displacement and the dipole become~~s~~ time dependent:

$$\bar{p}(t) = e \cdot \bar{x}(t) = \frac{e^2E_o/m_e}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}e^{i(\omega t \pm \varphi)}$$

The polarizability is defined as the induced polarization per electric field, that is to say:

$$\alpha \equiv \frac{p(t)}{E(t)} = \frac{e^2/m_e}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}$$

In a simple LIH medium, for the sake of simplicity, we assume that the dipoles follow the electric field and the overall **susceptibility** χ_e of a unit volume containing N atoms would be N times the atomic polarizability:

$$\chi_e = N\alpha = \frac{Np(t)}{E(t)} = \frac{Ne^2/m_e}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} = \frac{P(t)}{E(t)}$$

Where $P(t)$ is the **macroscopic** electrical polarization of the medium.

Next, remembering that $\frac{P(t)}{E(t)} = \varepsilon - \varepsilon_0$

$$\boxed{\begin{gathered}\varepsilon_0\bar{E} = \bar{D} - \bar{P} = \varepsilon\bar{E} - \bar{P} \Longrightarrow \\ \Longrightarrow \bar{P} = (\varepsilon - \varepsilon_0)\bar{E}\end{gathered}}$$

$$permittivity\ \varepsilon = \varepsilon_0 + \frac{P(t)}{E(t)} = \varepsilon_0 + \frac{Ne^2/m_e}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} \Longrightarrow n^2 = \varepsilon_r \equiv \frac{\varepsilon}{\varepsilon_0} = 1 + \frac{Ne^2}{\varepsilon_0 m_e[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}} \Longrightarrow$$

$$n = \sqrt{1 + \frac{Ne^2}{\varepsilon_0 m_e[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}}$$

$$m\frac{d^2x}{dt} = F_{restoring} + F_{damping} + F_{driving}, \qquad and\ idntroducing\ 1, 2, 3\ we\ get:$$

$$\boldsymbol{m\frac{d^2x}{dt} + m\gamma\dot{x} + m\omega_0^2 x = qEe^{-i\omega t}} \qquad [\mathbf{20}]$$

Dividing by m we get:

$$\boldsymbol{\frac{d^2x}{dt} + \gamma\dot{x} + \omega_0^2 x = \frac{q}{m}Ee^{-i\omega t}}$$

$$x(t) = x_0 e^{-i(\omega t+\alpha)} \qquad [21]$$

To evaluate the amplitude x_0, we introduce [21] into [20]:

$x_0(i\omega)^2 e^{-i(\omega t+\alpha)} + \omega_0^2 x_0 e^{-i(\omega t+\alpha)} = \frac{q}{m}Ee^{-i\omega t}$ /dividing by $e^{-i(\omega t+\alpha)}$ one get: $x_0(\omega_0^2 - \omega^2) = \frac{q}{m}Ee^{i\alpha} \Longrightarrow$

$$\boldsymbol{x_0 = \frac{qEe^{i\alpha}}{m(\omega_0^2 - \omega^2)}}$$