EMF-Chapter 4

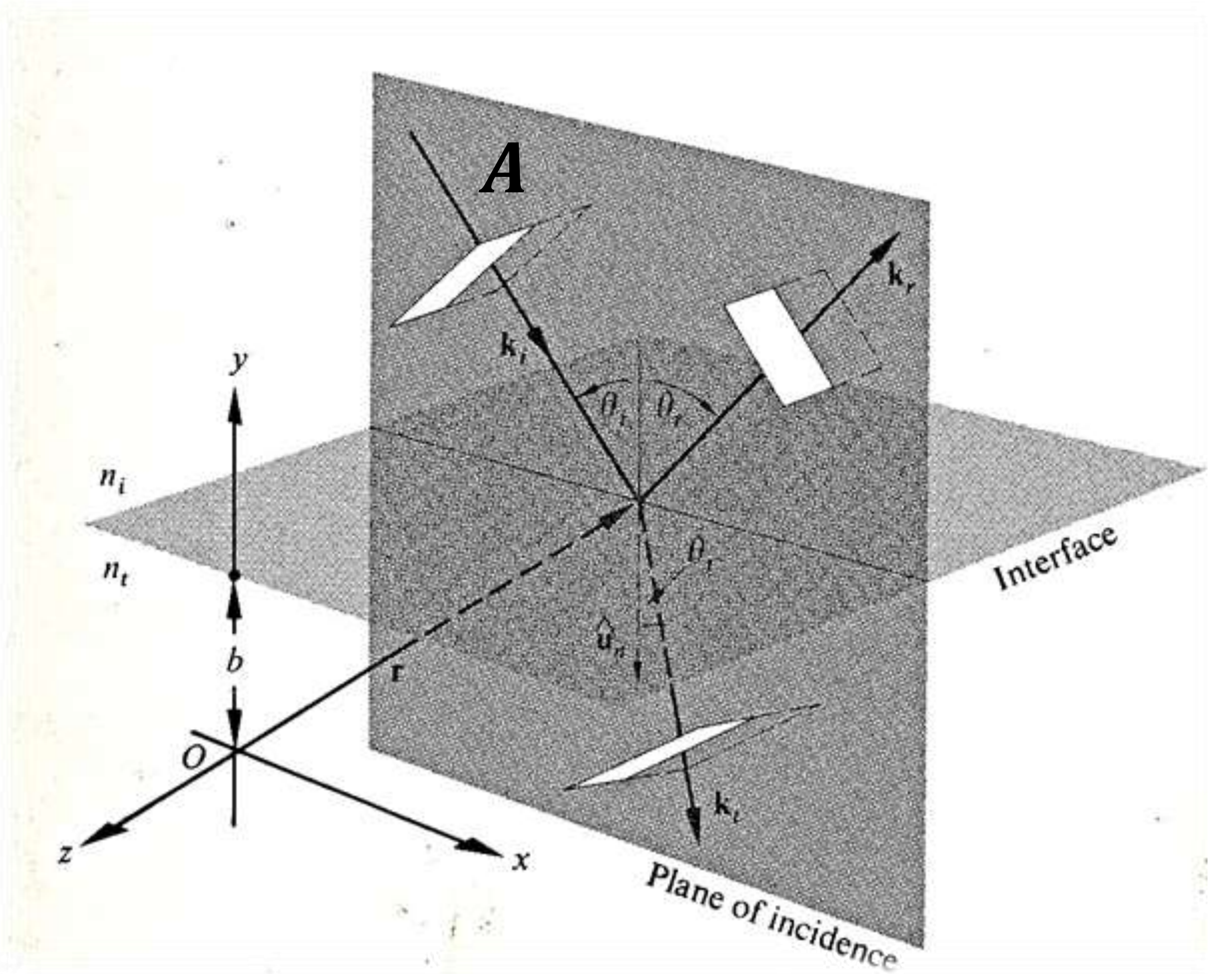
Reflection and Refraction

(Hecht 3rd edition)

There are a few ways to treat reflection/refraction but EMF theory considered to be the far more complete description

Given are two homogeneous lossless dialectic media, having n_i and n_t with a planar interface in between. We arbitrarily choose the origin to coincide with that of the Cartesian coordinate system.

We consider a planar monochromatic incident wave having the form: $\boldsymbol{\overline{E}_i = \overline{E}_{0i}\cos(\overline{k}_i \cdot \overline{r} - \omega_i t)}$ travels within the plane A located at z_0. This plane is perpendicular to the interface plane and parallel to the $y - x$ plane and hence, $\boldsymbol{\overline{k}_i}$ has $no\ z\ component$. The beam with $\overline{k}_i$ hits the interface plane at point o, at an incident angle θ_i in respect to $\widehat{u}$, unit~~e~~ vector normal to the interface at the point of incidence.



The reflected beam ($\overline{k}_r$) is redirected at angle θ_r back to the medium n_i in respect to $\widehat{u}$ and is not a priory assumed to be within plane A (see Figure 4.36).

Figure 4.36: Plane waves incident on the boundary between two homogeneous, isotropic, lossless dielectric media (He~~a~~cht 3rd Edition, p110).

Thus making no assumptions about the directions, frequencies, wavelengths, phases or amplitudes, the reflected and transmitted (refracted) waves can be expressed as:

$$\bar{\boldsymbol{E}}_{\boldsymbol{r}} = \bar{\boldsymbol{E}}_{\boldsymbol{0r}}\mathbf{cos}(\bar{\boldsymbol{k}}_{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{r}}\boldsymbol{t} + \boldsymbol{\varphi}_{\boldsymbol{r}}) \quad \textit{and} \quad \bar{\boldsymbol{E}}_{\boldsymbol{t}} = \bar{\boldsymbol{E}}_{\boldsymbol{0t}}\mathbf{cos}(\bar{\boldsymbol{k}}_{\boldsymbol{t}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{t}}\boldsymbol{t} + \boldsymbol{\varphi}_{\boldsymbol{t}})$$

Where φ_r and φ_t are phase constants relative to $\bar{E}_i$ due to the fact that the origin is not unique. Had the origin been placed in the incident point, then $\varphi_r = \varphi_t = 0$.

Next, boundary condition in EMF dictates that the tangential components of the electric fields across the two sides of the interface must be **continuous**. These tangential components, regardless the direction of $\bar{E}$ can be determined from the cross-product with $\hat{u}$, that is to say:

$$\hat{u} \times \bar{\boldsymbol{E}}_{\boldsymbol{i}} + \hat{u} \times \bar{\boldsymbol{E}}_{\boldsymbol{r}} = \hat{u} \times \bar{\boldsymbol{E}}_{\boldsymbol{t}}$$

Or

$$\hat{u} \times \bar{\boldsymbol{E}}_{\boldsymbol{0i}}\mathbf{cos}(\bar{\boldsymbol{k}}_{\boldsymbol{i}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{i}}\boldsymbol{t}) + \hat{u} \times \bar{\boldsymbol{E}}_{\boldsymbol{0r}}\mathbf{cos}(\bar{\boldsymbol{k}}_{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{r}}\boldsymbol{t}) = \hat{u} \times \bar{\boldsymbol{E}}_{\boldsymbol{0t}}\mathbf{cos}(\bar{\boldsymbol{k}}_{\boldsymbol{t}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{t}}\boldsymbol{t})$$

This equality must exist independent of $t \textit{ and } r$ and hence the EMF $E_i, E_r, \textit{and } E_t$ must have precisely the same functional dependence on the variables $t \textit{ and } r, \textit{which means that}$:

$$\left|(\bar{\boldsymbol{k}}_{\boldsymbol{i}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{i}}\boldsymbol{t})\right|_{\boldsymbol{y=b}} = \left|(\bar{\boldsymbol{k}}_{\boldsymbol{r}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{r}}\boldsymbol{t} + \boldsymbol{\varphi}_{\boldsymbol{r}})\right|_{\boldsymbol{y=b}} = \left|(\bar{\boldsymbol{k}}_{\boldsymbol{t}} \cdot \bar{\boldsymbol{r}} - \boldsymbol{\omega}_{\boldsymbol{t}}\boldsymbol{t} + \boldsymbol{\varphi}_{\boldsymbol{t}})\right|_{\boldsymbol{y=b}}$$

Since this equality holds for any t and r, their coefficients must be equal, i.e.:

$$\boldsymbol{\omega_i = \omega_r = \omega_t}$$

Which is in agreement with the fact that the electrons within the medium undergoing forced vibrations at the frequency of the incident (driving) wave. Similarly:

$$\left|(\overline{\boldsymbol{k}}_{\boldsymbol{i}} \cdot \overline{\boldsymbol{r}})\right|_{\boldsymbol{y=b}} = \left|(\overline{\boldsymbol{k}}_{\boldsymbol{r}} \cdot \overline{\boldsymbol{r}} + \boldsymbol{\varphi_r})\right|_{\boldsymbol{y=b}} = \left|(\overline{\boldsymbol{k}}_{\boldsymbol{t}} \cdot \overline{\boldsymbol{r}} + \boldsymbol{\varphi_t})\right|_{\boldsymbol{y=b}}$$ where $\overline{\boldsymbol{r}}$ remains on the interface

From the left two terms we obtain: $\left[(\overline{\boldsymbol{k}}_{\boldsymbol{i}} - \overline{\boldsymbol{k}}_{\boldsymbol{r}}) \cdot \overline{\boldsymbol{r}}\right]_{\boldsymbol{y=b}} = \boldsymbol{\varphi_r}$

We previously proved that the scalar product $\overline{\boldsymbol{k}} \cdot \overline{\boldsymbol{r}} = \boldsymbol{constant}$ $describes\ a\ plane\ to\ which\ \overline{k}$ is perpendicular and is sweeps out by the end point of $\overline{\boldsymbol{r}}$. Hence, the vector $(\overline{\boldsymbol{k}}_{\boldsymbol{i}} - \overline{\boldsymbol{k}}_{\boldsymbol{r}})$

1. is perpendicular to the interface plane.
2. is parallel to $\widehat{\boldsymbol{u}}$.
3. $k_i = k_r$ since the waves, having the vectors $\overline{\boldsymbol{k}}_{\boldsymbol{i}}$ and $\overline{\boldsymbol{k}}_{\boldsymbol{r}}$, are in the same medium,
4. has no component in the interface plane, i.e. parallel to $\widehat{\boldsymbol{u}} \Longrightarrow$ $\overline{\boldsymbol{k}}_{\boldsymbol{i}}$, $\overline{\boldsymbol{k}}_{\boldsymbol{r}}$ and $\widehat{\boldsymbol{u}}$ $are\ in\ the\ plane$ $\boldsymbol{A}$ and hence:
5. $\widehat{\boldsymbol{u}} \times \left(\overline{\boldsymbol{k}}_{\boldsymbol{i}} - \overline{\boldsymbol{k}}_{\boldsymbol{r}}\right) = 0 \quad \Longrightarrow \quad k_i sin\theta_i = k_r sin\theta_r \Longrightarrow$

Law of reflection: $$\boldsymbol{\theta_i = \theta_r}$$

Next, from the left and last terms of: $\left|(\overline{\boldsymbol{k}}_i \cdot \overline{\boldsymbol{r}} - \boldsymbol{\omega}_i \boldsymbol{t})\right|_{y=b} = \left|(\overline{\boldsymbol{k}}_r \cdot \overline{\boldsymbol{r}} - \boldsymbol{\omega}_r \boldsymbol{t} + \boldsymbol{\varphi}_r)\right|_{y=b} = \left|(\overline{\boldsymbol{k}}_t \cdot \overline{\boldsymbol{r}} - \boldsymbol{\omega}_t \boldsymbol{t} + \boldsymbol{\varphi}_t)\right|_{y=b}$ one gets:

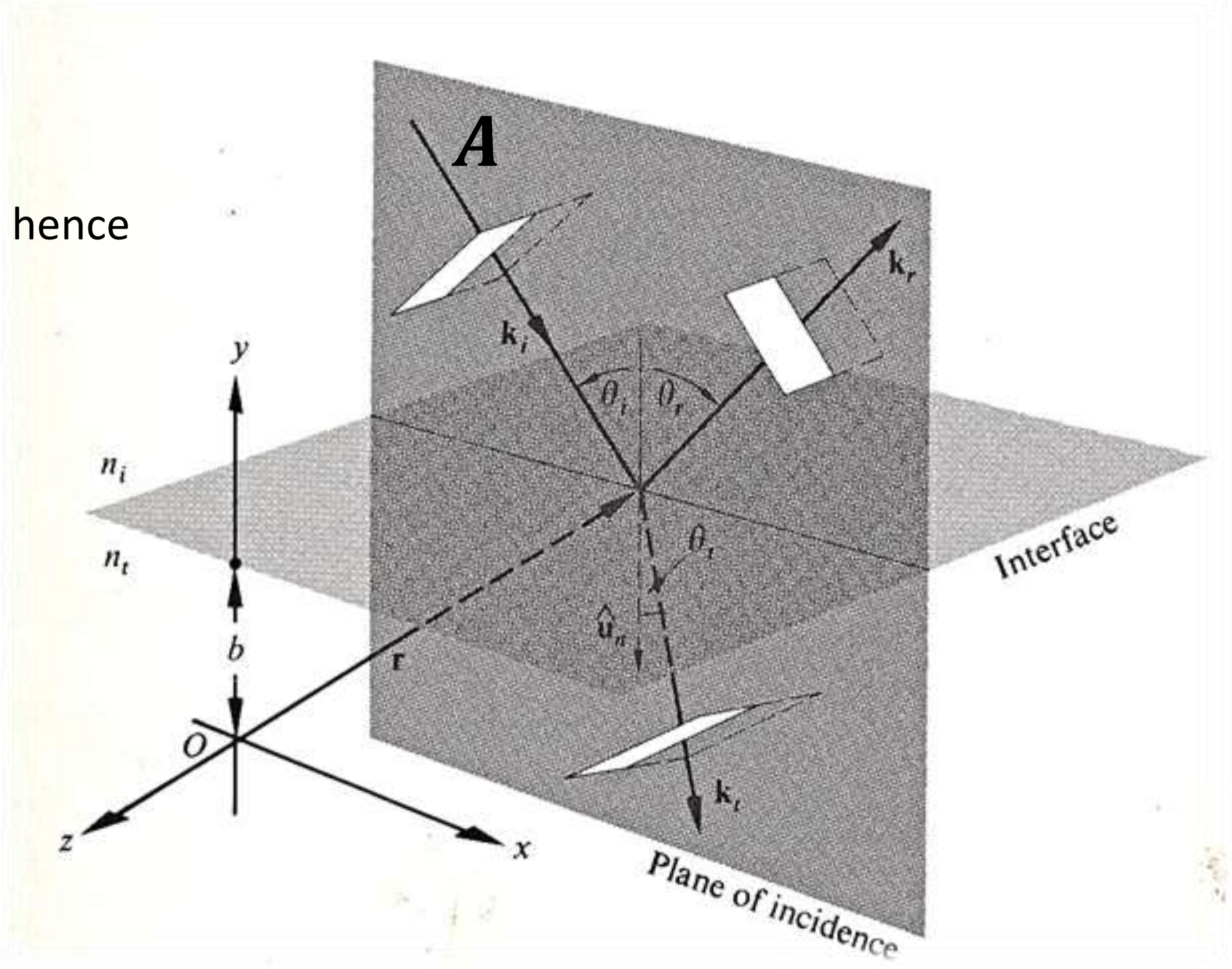
$$\left[(\overline{\boldsymbol{k}}_i - \overline{\boldsymbol{k}}_t) \cdot \overline{\boldsymbol{r}}\right]_{y=b} = \boldsymbol{\varphi}_t \quad \Longrightarrow$$

$(\overline{\boldsymbol{k}}_i - \overline{\boldsymbol{k}}_t) \perp$ ***to the interface plane*** and hence
$\overline{\boldsymbol{k}}_i,\ \overline{\boldsymbol{k}}_r,\ \overline{\boldsymbol{k}}_t$ *and* $\widehat{\boldsymbol{u}}$ *are coplanar*, i.e. within *the plane* ***A*** and hence

$$\widehat{\boldsymbol{u}} \times (\overline{\boldsymbol{k}}_i - \overline{\boldsymbol{k}}_t) = 0 \quad \Longrightarrow \quad k_i sin\theta_i = k_t sin\theta_t \quad [24]$$

Introducing $k_j = \frac{\omega}{v_j} = \frac{\omega}{c/n_j}$ *into* [24] *yields* ***Snell's law***:

$$\boldsymbol{n_i sin\theta_i = n_t sin\theta_t}$$



Fresnel equations

Fresnel Equations:

For any polarization of the incident, reflected and transmitted (refracted) waves, their electric field can be separated into the components which vibrate normal (s) and parallel (p) (within) to the plane of incidence.

The relevant $\bar{E}$ and $\bar{B}$ fields $and\ \bar{k}_s$ are depicted in the upper panel of Figure 4.37. In the following, the ratios between the amplitudes will be analyzed, subject to the boundary conditions discussed above.

Case 1: $\bar{E}$ *is* normal $(s\ component)$ to (and $\bar{B}$ within) the plane of incidence:

Recall that: $E = vB\ and\ that\ \bar{E} \perp \bar{B} \perp \bar{k},\ than$:

$$\hat{k} \times \bar{E} = v\bar{B}$$

Due to the continuity of the tangential components of the E-field, we have everywhere at the boundary at any time point (we arbitrarily choose that $E_i, E_r,\ and\ E_t$ are all directed towards the reader at the interface) and remembering that at the interface all cosines are equal to one):

$$(s\ components)\ \ E_{0i} + E_{0r} = E_{0t} \quad [25]$$

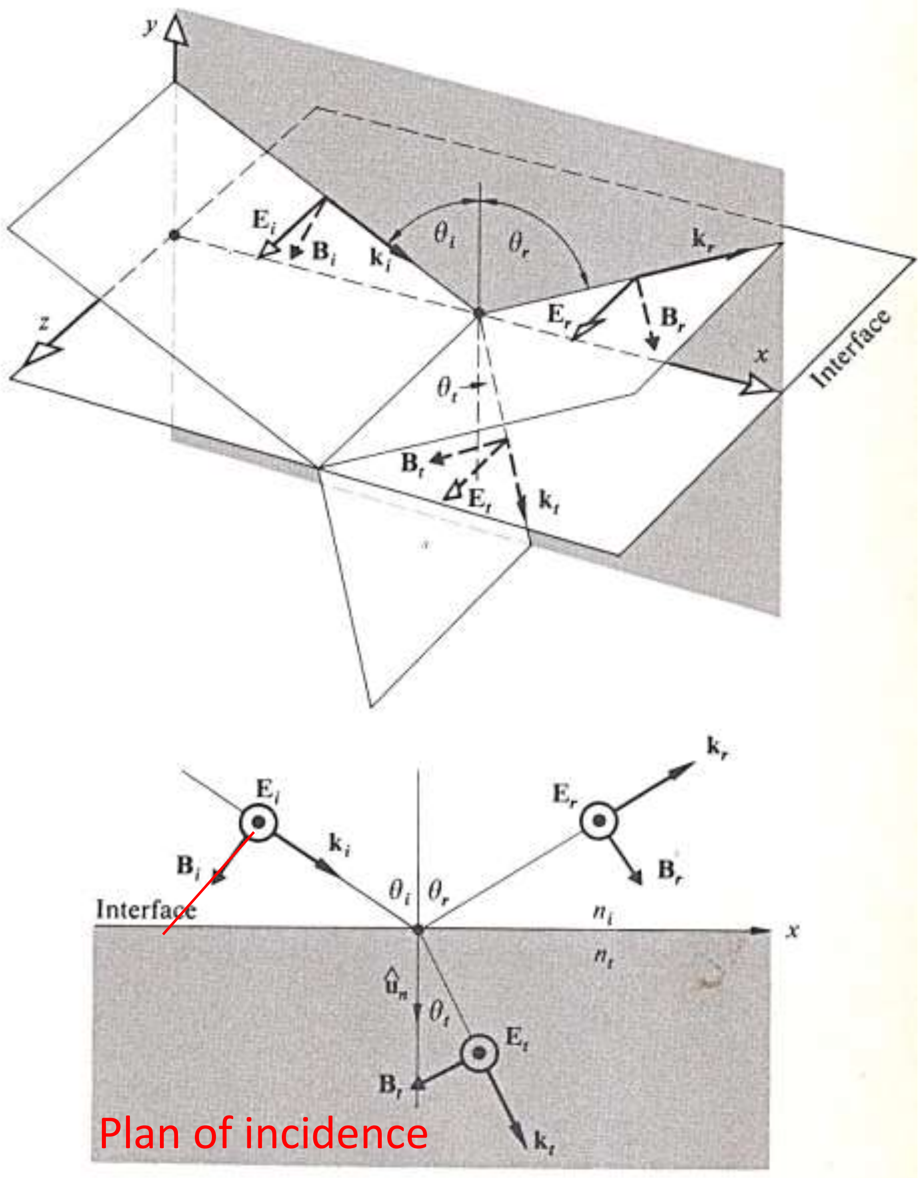


FIGURE 4.37 An incoming wave whose E-field is normal to the plane-of-incidence. (Hecht)

Recalling the boundary conditions for the tangential (p) components of H, B:

For $i_f = 0, \quad \Delta H_T = 0 \quad \Longrightarrow \frac{B_{T1}}{\mu_1} = \frac{B_{T2}}{\mu_2}$

$$-H_i cos\theta_i + H_r cos\theta_r = -H_t cos\theta_t \quad (+, - \text{ are in respect to the increasing x})$$

In biological and dielectric media: $\mu_i = \mu_t \cong \mu_0$ and recalling that: $H = \frac{B}{\mu} = \frac{1}{\mu}\frac{E}{v} = \frac{1}{\mu}\frac{E}{c/n} = \frac{nE}{\mu_0 c}$

and introducing $H = \frac{nE}{\mu_0 c}$ in [26] and realizing that: $\theta_i = \theta_r \ and \ n_i = n_r$ one gets for the $s \ components \ of \ E$:

$$n_i(E_{0i} - E_{0r})cos\theta_i = n_t E_{0t} cos\theta_t \qquad [26]$$

$and \ togaether \ with$ [25], $i.e. \mathbf{E_{0i}} + \mathbf{E_{0r}} = \mathbf{E_{0t}}$, one constraucts the following $Fresnel$ amplitude $ratios$:

$$\boldsymbol{r_s = r_\perp \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_s = \frac{n_i cos\theta_i - n_t cos\theta_t}{n_i cos\theta_i + n_t cos\theta_t}} \quad and$$

$$\boldsymbol{t_s = t_\perp \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_s = \frac{2 n_i cos\theta_i}{n_i cos\theta_i + n_t cos\theta_t}}$$

Where, $r_\perp \ and \ t_\perp$ denote~~s~~ the ***s amplitude reflection and transmission*** *(refraction)* ***coefficients*** respectively.

Case 2: $\bar{E}$ *is* parallel $(p\ component)$ to (and $\bar{B}$ normal) the plane of incidence:

Again, due to the continuity of the tangential components of the E-field, we have (see Figure 4.38):

$$(p)\ \ E_{0i}cos\theta_i - E_{0r}cos\theta_r = E_{0t}cos\theta_t \qquad [27]$$

$$(p;\ \theta_i = \theta_r)\ \ (E_{0i} - E_{0r})cos\theta_r = E_{0t}cos\theta_t \qquad [28]$$

$$From\ the\ continuaity\ of\ H_p:\quad H_i + H_r = H_t \qquad [29]$$

$$Again, recalling\ the\ relation:\ H = \frac{B}{\mu} = \frac{1}{\mu}\frac{E}{v} = \frac{1}{\mu}\frac{E}{c/n} = \frac{nE}{\mu_0 c}$$

and substituting into [29] one gets :

$$n_i(E_{0i} + E_{0r}) = n_t E_{0t} \qquad [30]$$

Dividing [28] and [30] by E_{0i} , *extracting* $\frac{E_{0r}}{E_{0i}}$ *and* $\frac{E_{0t}}{E_{0i}}$ *and rearrange both equations, one gets*:

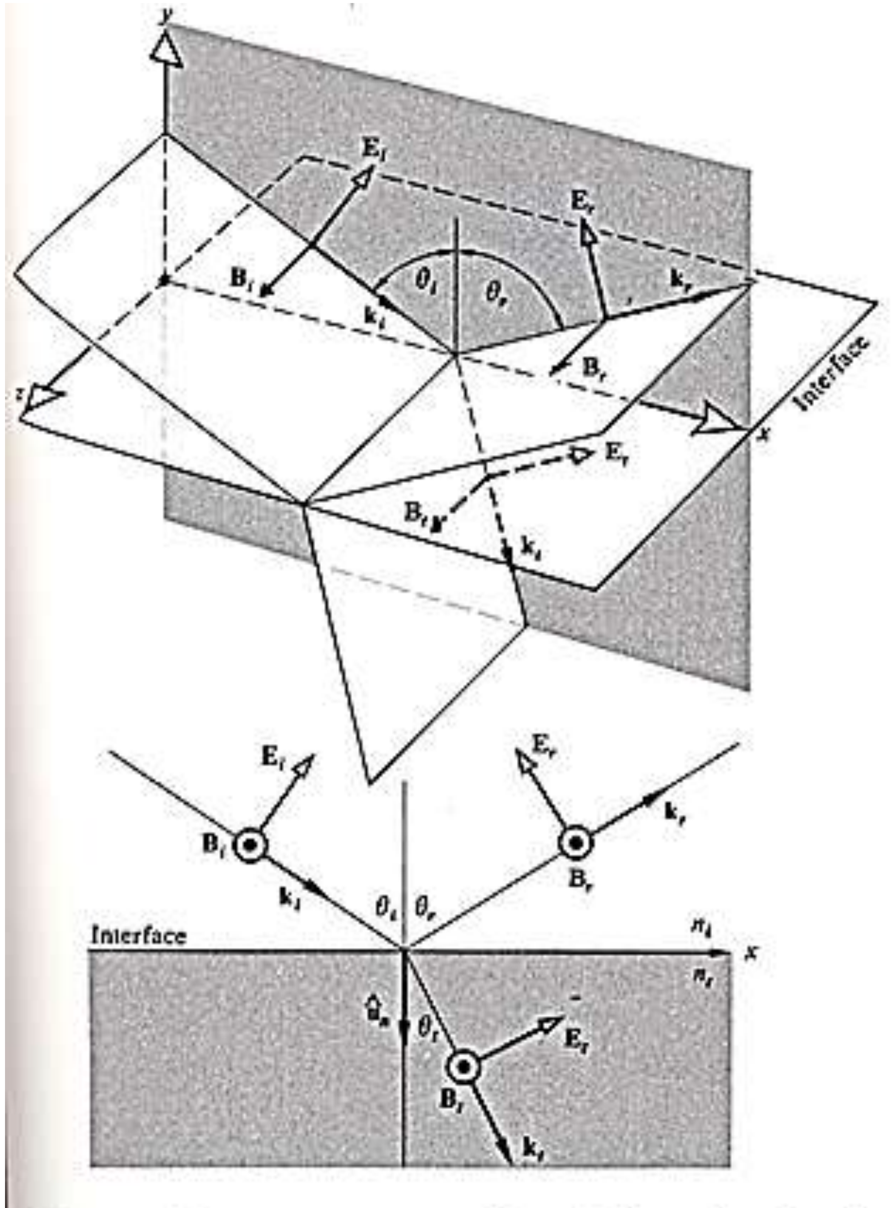


FIGURE 4.38 An incoming wave whose E-field is in the plane-of-incidence.

$$r_p = r_\parallel \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_p = \frac{n_t cos\theta_i - n_i cos\theta_t}{n_i cos\theta_t + n_t cos\theta_i} \quad and \quad t_p = t_\parallel \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_p = \frac{2n_i cos\theta_i}{n_i cos\theta_t + n_t cos\theta_i}$$

Where, $r_\parallel$ *and* $t_\parallel$ denotes the ***p amplitude reflection and transmission*** *(refraction)* ***coefficients*** respectively.

The Fresnel coefficients:

$$r_s = r_\perp \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_s = \frac{n_i cos\theta_i - n_t cos\theta_t}{n_i cos\theta_i + n_t cos\theta_t} \qquad t_s = t_\perp \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_s = \frac{2n_i cos\theta_i}{n_i cos\theta_i + n_t cos\theta_t}$$

$$r_p = r_\parallel \equiv \left(\frac{E_{0r}}{E_{0i}}\right)_p = \frac{n_t cos\theta_i - n_i cos\theta_t}{n_i cos\theta_t + n_t cos\theta_i} \qquad t_p = t_\parallel \equiv \left(\frac{E_{0t}}{E_{0i}}\right)_p = \frac{2n_i cos\theta_i}{n_i cos\theta_t + n_t cos\theta_i}$$

Some practical aspects of Fresnel equations:

Using Snell's law the above 4 refractive index-based equations can be re-expressed by $\boldsymbol{\theta_i}$ and $\boldsymbol{\theta_t}$:

$$r_\perp = -\frac{sin(\theta_i - \theta_t)}{sin(\theta_i + \theta_t)} \quad [a] \qquad r_\parallel = +\frac{tan(\theta_i - \theta_t)}{tan(\theta_i + \theta_t)} \quad [b]$$

$$t_\perp = +\frac{2sin\theta_t cos\theta_i}{sin(\theta_i + \theta_t)} \quad [c] \qquad t_\parallel = +\frac{2sin\theta_t cos\theta_i}{sin(\theta_i + \theta_t)cos(\theta_i - \theta_t)} \quad [d]$$

1. For $\boldsymbol{\theta_i} \rightarrow \mathbf{0}$ (normal incidence) the **tangents** in [b] → **sines** i.e. $[\,\boldsymbol{r_\parallel}]_{\boldsymbol{\theta_i \rightarrow 0}} = \left|\frac{sin(\theta_i-\theta_t)}{sin(\theta_i+\theta_t)}\right|_{\theta_i \rightarrow 0} = -[\,\boldsymbol{r_\perp}]_{\boldsymbol{\theta_i \rightarrow 0}}$.
2. The equality $[\,r_\parallel]_{\theta_i \rightarrow 0} = -[\,r_\perp]_{\theta_i \rightarrow 0}$ is a result of unspecified plane of incident in such a scenario.
3. Expanding the sines of $[1]$ *and employing Snell's law, yields*: $[\,r_\parallel]_{\theta_i \rightarrow 0} = -[\,r_\perp]_{\theta_i \rightarrow 0} = \frac{n_t cos\theta_i - n_i cos\theta_t}{n_t cos\theta_i + n_i cos\theta_t}$
4. since $\boldsymbol{\theta}_\mathrm{i} \rightarrow \mathbf{0} \implies \boldsymbol{\theta_t} \rightarrow \mathbf{0}$ *as well*, then $[\,r_\parallel]_{\theta_i \rightarrow 0} = -[\,r_\perp]_{\theta_i \rightarrow 0} = \frac{n_t cos\theta_i - n_i cos\theta_t}{n_t cos\theta_i + n_i cos\theta_t}$ *becomes* $\frac{n_t - n_i}{n_t + n_i}$

For instance: at air (n_i=1)-glass (n_t=1.5) interface $[\,\boldsymbol{r_\parallel}]_{\boldsymbol{\theta_i \rightarrow 0}} = -[\,\boldsymbol{r_\perp}]_{\boldsymbol{\theta_i \rightarrow 0}} = \pm\mathbf{0.2}$.

Snell's law teaches that for $n_t > n_i$, $\theta_t < \theta_i$ and $r_\perp(r_s) < 0$ $for\ all\ \theta_i\ (see\ Figure\ 4.39)$.

Brewster's (polarization) angle:

At $(\boldsymbol{\theta_i + \theta_t}) = \frac{\boldsymbol{\pi}}{\mathbf{2}}$ $\boldsymbol{tan(\theta_i + \theta_t) = \infty\ and}$

Equation [b]: $r_\parallel = +\frac{tan(\theta_i - \theta_t)}{tan(\theta_i + \theta_t)} \longrightarrow \mathbf{0}$ ($\boldsymbol{see}$ Fig. 4.39),

for which $\theta_i \equiv \theta_B \equiv \theta_p$ (p-polarization).

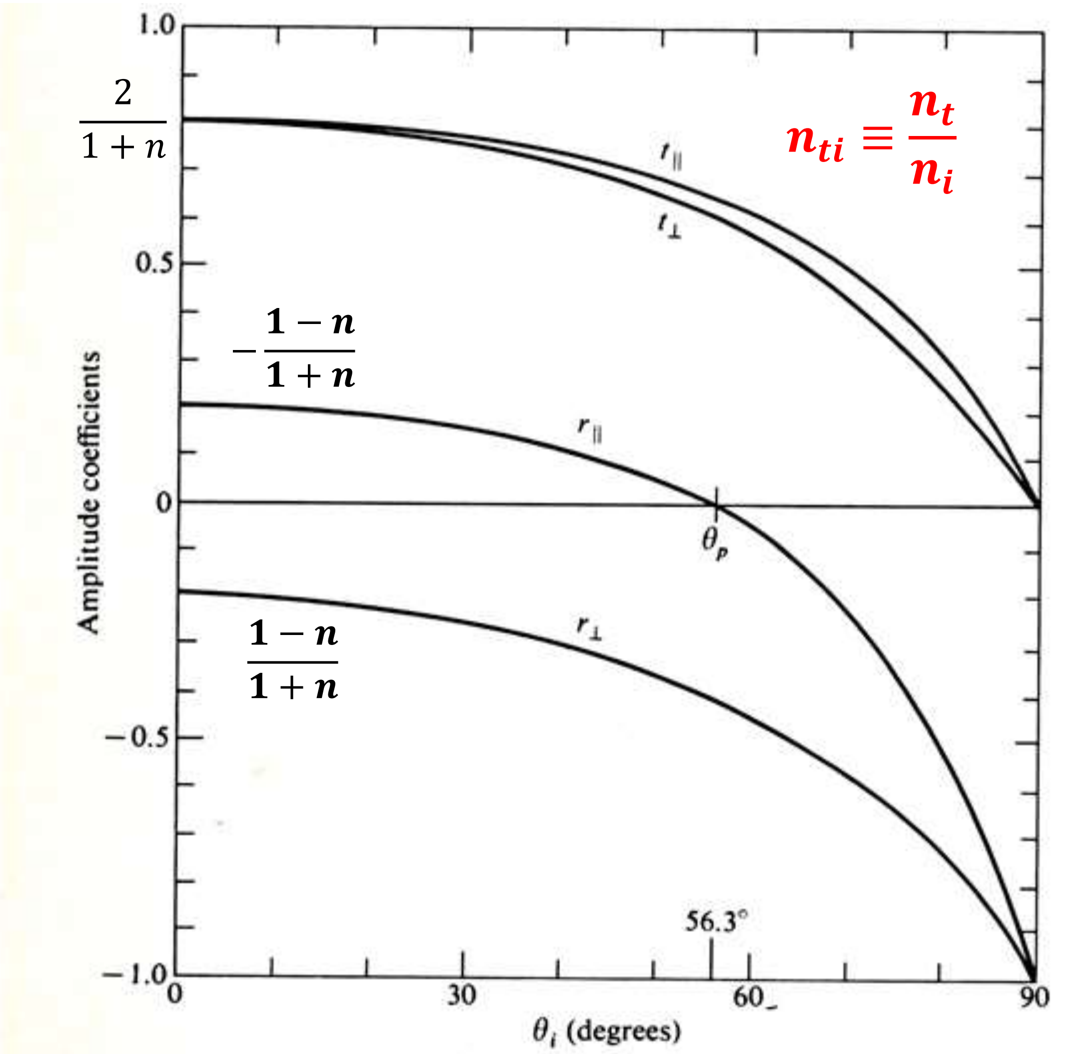


FIGURE 4.39 The amplitude coefficients of reflection and transmission as a function of incident angle. These correspond to external reflection $n_t > n_i$ at an air–glass interface ($n_{ti} = 1.5$).

At the Brewster angle θ_B, the reflected wave is totally $s - polarized, a\ fact\ which\ makes\ this\ a\ way\ to\ polarize\ light\ by\ reflection$.

At $(\boldsymbol{\theta_i + \theta_t}) = \frac{\pi}{2} \rightarrow \boldsymbol{\theta_t} = \frac{\pi}{2} - \boldsymbol{\theta_i} \Rightarrow$

$$n_i sin\theta_B = n_t sin\left(\frac{\pi}{2} - \theta_B\right) = n_t cos\theta_B \Longrightarrow$$

$$\Longrightarrow \boldsymbol{tan\theta_B = \frac{n_t}{n_i} = n_{ti}}$$

Reflectance and Transmittance

Recalling that (a) the power density (i.e. per unite surface of a beam cross-section) is given by **the Poynting vector** $\bar{N} = c^2\varepsilon_0\bar{E}\times\bar{B}$ and that (b) its intensity I - **radiation flux density** (i.e. average energy per unite time crossing a unite area normal to $\bar{N}$, $\langle Wm^{-2}\rangle$) is

$$I = \langle N\rangle_T = \frac{c\varepsilon_0}{2}E_0^2$$

Regarding Figure 8.18: let $I_i, I_r, and\ I_t$ be the incident , reflected and the transmitted flux densities accordingly (beam intensity). As shown in the figure, the cross sections are $A\cos\theta_i, A\cos\theta_r, A\cos\theta_t$ and hence the power (the energy per unit time) of the incident, reflected and transmitted beams are $I_iA\cos\theta_i, I_rA\cos\theta_r, I_tA\cos\theta_t$

The **reflectance** ***R*** is defined as the ratio between the reflected and incident Powers (transported energy per unit time):

$$R \equiv \frac{I_rA\cos\theta_r}{I_iA\cos\theta_i} = \frac{I_r}{I_i} = \frac{\frac{v_r\varepsilon_r}{2}E_{0r}^2}{\frac{v_i\varepsilon_i}{2}E_{0i}^2} = \rightarrow same\ medium =$$

$$= \frac{E_{0r}^2}{E_{0i}^2} = r^2$$

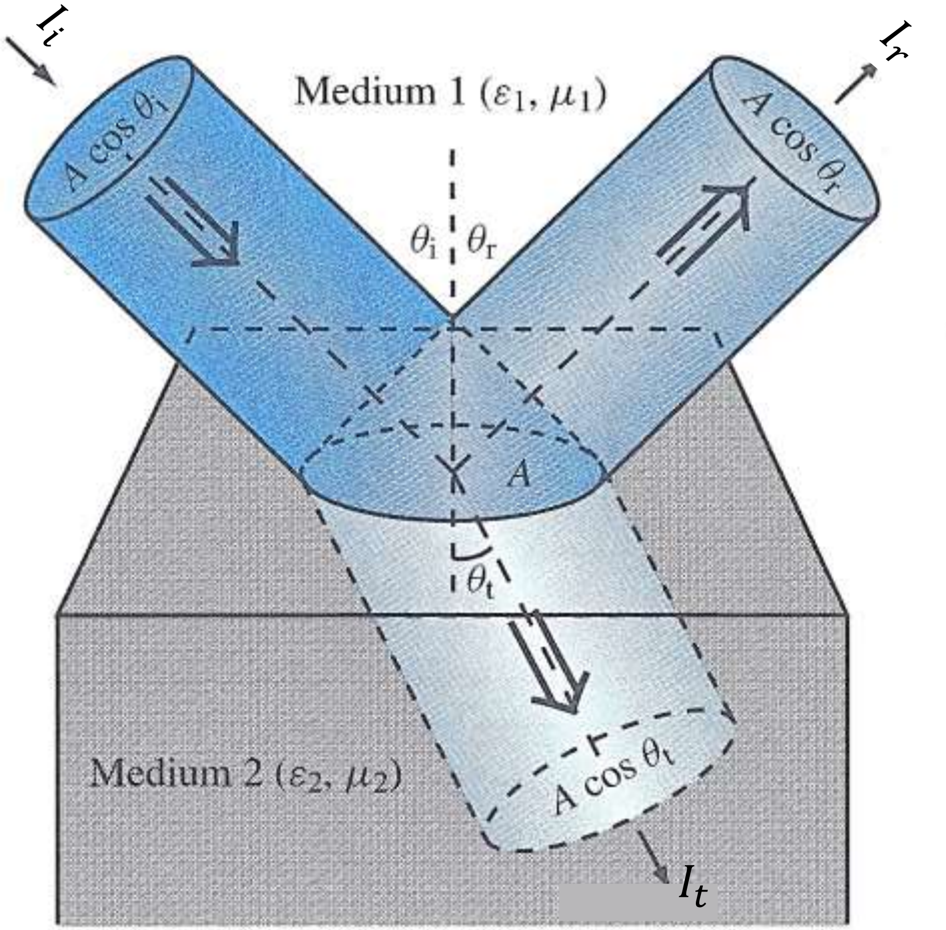


Figure 8-18: Reflection and transmission of an incident circular beam illuminating a spot of size A on the interface.