Dividing by ε_0 *to yield* ε^r

Similarly, ***trasmittance*** (T) *is defined as*:

$$T \equiv \frac{trasmitted\ power}{icident\ power} = \frac{I_t A cos\theta_t}{I_i A cos\theta_i} = \frac{\frac{\frac{v_t \varepsilon_t}{\varepsilon_0}}{2} E_{0t}^2}{\frac{\frac{v_i \varepsilon_i}{\varepsilon_0}}{2} E_{0i}^2} \cdot \frac{cos\theta_t}{cos\theta_i} = \frac{v_t \varepsilon_t^r E_{0t}^2}{v_i \varepsilon_i^r E_{0i}^2} \cdot \frac{cos\theta_t}{cos\theta_i} \qquad [40]$$

Next, recalling that $\boldsymbol{n^2 = \varepsilon_r \mu_r \cong \varepsilon_r}$ and that $v_j = \frac{c}{n_j}$ *and introducing it into* [40] *gives*:

$$T = \frac{v_t \varepsilon_t^r E_{0t}^2}{v_i \varepsilon_i^r E_{0i}^2} \cdot \frac{cos\theta_t}{cos\theta_i} = \frac{\frac{c}{n_t} n_t^2 E_{0t}^2}{\frac{c}{n_i} n_i^2 E_{0i}^2} \cdot \frac{cos\theta_t}{cos\theta_i} = \frac{n_t E_{0t}^2}{n_i E_{0i}^2} \cdot \frac{cos\theta_t}{cos\theta_i} = \frac{n_t cos\theta_t}{n_i cos\theta_i} \frac{E_{0t}^2}{E_{0i}^2} = \frac{n_t cos\theta_t}{n_i cos\theta_i} t^2$$

So in summary we have:

$$R = r^2$$

and

$$T = \frac{n_t cos\theta_t}{n_i cos\theta_i} t^2 = \frac{\frac{1}{v_t} cos\theta_t}{\frac{1}{v_i} cos\theta_i} t^2$$

As can be seen, $\boldsymbol{T}$, *unlike* $\boldsymbol{R} = \boldsymbol{r^2}$, *is not simply equal to* $\boldsymbol{t^2}$. There are two reasons for that:

1. Recalling that in a medium $\boldsymbol{I} = \langle \boldsymbol{N_T} \rangle_{\boldsymbol{t}} = \frac{\boldsymbol{v\varepsilon}}{\boldsymbol{2}} \boldsymbol{E_0^2} = \frac{\frac{\boldsymbol{c}}{\boldsymbol{n}}\boldsymbol{\varepsilon}}{\boldsymbol{2}} \boldsymbol{E_0^2}$, the higher the velocity the more energy is transported. On the other hand, the denser (optically) the medium, the slower the speed of energy transported through it. That is to say, the intensity is velocity dependent, i.e. $I \ \alpha \ v$, a fact which justifies the dependence of $\boldsymbol{T}$ on indexes of refraction.

2. The cross section area of the incident and reflected beams are the same, but different from that of the transmitted beam, a fact which is manifested by the cosines ratio.

Conservation of energy

$$Energy|_{\,i} = Energy|_{\,r} + Energy|_{\,t} = PD_{\,i} \cdot A_i = PD_{\,r} \cdot A_r + PD_{\,t} \cdot A_t =$$

$$\frac{\boldsymbol{v_i \varepsilon_i}}{\boldsymbol{2}} \boldsymbol{E_{0i}^2 cos\theta_i} = \frac{\boldsymbol{v_{r=i}\varepsilon_{r=i}}}{\boldsymbol{2}} \boldsymbol{E_{0r}^2 cos\theta_{r=i}} + \frac{\boldsymbol{v_t \varepsilon_t}}{\boldsymbol{2}} \boldsymbol{E_{0t}^2 cos\theta_t} \qquad [41]$$

Dividing [41] by its first left term yields:

$$1 = \frac{\boldsymbol{E_{0r}^2}}{\boldsymbol{E_{0i}^2}} + \frac{v_t}{v_i} \cdot \frac{\cancel{\mu_0}\boldsymbol{\varepsilon_t}}{\cancel{\mu_0}\boldsymbol{\varepsilon_i}} \cdot \frac{\boldsymbol{cos\theta_t}}{\boldsymbol{cos\theta_i}} \frac{\boldsymbol{E_{0t}^2}}{\boldsymbol{E_{0i}^2}} = \frac{\boldsymbol{E_{0r}^2}}{\boldsymbol{E_{0i}^2}} + \frac{v_t}{v_i} \frac{(1/v_t{}^2)}{(1/v_i{}^2)} \cdot \frac{\boldsymbol{cos\theta_t}}{\boldsymbol{cos\theta_i}} \frac{\boldsymbol{E_{0t}^2}}{\boldsymbol{E_{0i}^2}} = \frac{\boldsymbol{E_{0r}^2}}{\boldsymbol{E_{0i}^2}} + \frac{n_t \boldsymbol{cos\theta_t}}{n_i \boldsymbol{cos\theta_i}} \frac{\boldsymbol{E_{0t}^2}}{\boldsymbol{E_{0i}^2}} \rightarrow$$

$$\rightarrow \boldsymbol{1 = r^2 + \frac{n_t cos\theta_t}{n_i cos\theta_i} t^2 = R + T} \quad \boldsymbol{where\ there\ is\ no\ absorption}$$

Recalling that for $\boldsymbol{\theta}_{\mathrm{i}} = \mathbf{0} \implies \boldsymbol{\theta}_t = \mathbf{0}$

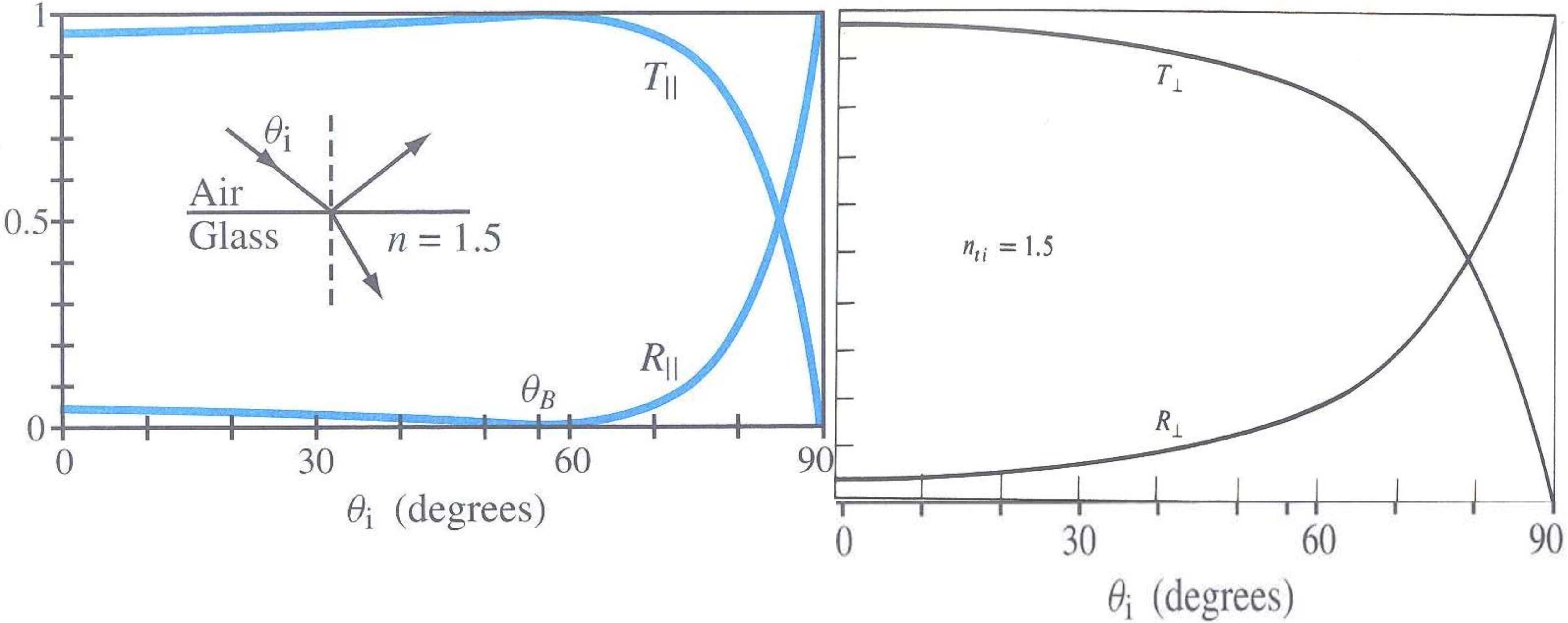
1. $[\,\mathrm{r}_{\parallel}]_{\theta_{\mathrm{i}}=0} = -[\,\mathrm{r}_{\perp}]_{\theta_{\mathrm{i}}=0} = \left.\dfrac{\boldsymbol{n_t cos\theta_i - n_i cos\theta_t}}{\boldsymbol{n_t cos\theta_i + n_i cos\theta_t}}\right|_{\boldsymbol{\theta_i=0}} = \dfrac{\boldsymbol{n_t - n_i}}{\boldsymbol{n_t + n_i}}$ and similarly*:*

2. $[\,\mathrm{t}_{\parallel}]_{\theta_{\mathrm{i}}=0} = [\,\mathrm{t}_{\perp}]_{\theta_{\mathrm{i}}=0} = \left.\dfrac{\boldsymbol{2n_i cos\theta_i}}{\boldsymbol{n_i cos\theta_i + n_t cos\theta_t}}\right|_{\boldsymbol{\theta_i=0}} = \dfrac{\boldsymbol{2n_i}}{\boldsymbol{n_i + n_t}}$

Then:

$$R = \mathrm{r}^2 = R_{\parallel} = R_{\perp} = \left(\frac{n_t - n_i}{n_t + n_i}\right)^2 \qquad \text{and} \qquad T = T_{\parallel} = T_{\perp} = \left(\frac{2n_i}{n_t + n_i}\right)^2$$

Hence, for external reflection, 4% of the light traveling in air and normally hitting a glass plane will be reflected back to air.



Ulaby &Hecht: Angular plots for $R_\|$, $T_\|$ $(left)$ and $R_\perp$, $T_\perp$ $(right)$ $as\ a\ function\ of$ $\boldsymbol{\theta}_\mathrm{i}$ for an air-glass interface.

Note: (a) For any $\boldsymbol{\theta}_\mathrm{i}$ the sums $R_\| + T_\|$ and $R_\perp + T_\perp$ $are\ each\ equals\ unity$ (1) and (b) at the Brewster angle $\theta_{\boldsymbol{B}}$, $R_\| = 0$ and $T_\|$=1.

Table 8-2: Expressions for Γ, τ, R, and T for wave incidence from a medium with intrinsic impedance η_1 onto a medium with intrinsic impedance η_2. Angles θ_i and θ_t are the angles of incidence and transmission, respectively.

Property	Normal Incidence $\theta_\mathrm{i} = \theta_\mathrm{t} = 0$	Perpendicular Polarization	Parallel Polarization
Reflection coefficient	$\Gamma = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$	$\Gamma_\perp = \dfrac{\eta_2 \cos\theta_\mathrm{i} - \eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}$	$\Gamma_\parallel = \dfrac{\eta_2 \cos\theta_\mathrm{t} - \eta_1 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}}$
Transmission coefficient	$\tau = \dfrac{2\eta_2}{\eta_2 + \eta_1}$	$\tau_\perp = \dfrac{2\eta_2 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{i} + \eta_1 \cos\theta_\mathrm{t}}$	$\tau_\parallel = \dfrac{2\eta_2 \cos\theta_\mathrm{i}}{\eta_2 \cos\theta_\mathrm{t} + \eta_1 \cos\theta_\mathrm{i}}$
Relation of Γ to τ	$\tau = 1 + \Gamma$	$\tau_\perp = 1 + \Gamma_\perp$	$\tau_\parallel = (1 + \Gamma_\parallel) \dfrac{\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{t}}$
Reflectivity	$R = \lvert\Gamma\rvert^2$	$R_\perp = \lvert\Gamma_\perp\rvert^2$	$R_\parallel = \lvert\Gamma_\parallel\rvert^2$
Transmissivity	$T = \lvert\tau\rvert^2 \left(\dfrac{\eta_1}{\eta_2}\right)$	$T_\perp = \lvert\tau_\perp\rvert^2 \dfrac{\eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i}}$	$T_\parallel = \lvert\tau_\parallel\rvert^2 \dfrac{\eta_1 \cos\theta_\mathrm{t}}{\eta_2 \cos\theta_\mathrm{i}}$
Relation of R to T	$T = 1 - R$	$T_\perp = 1 - R_\perp$	$T_\parallel = 1 - R_\parallel$
Notes: (1) $\sin\theta_\mathrm{t} = \sqrt{\mu_1\varepsilon_1/\mu_2\varepsilon_2}\,\sin\theta_\mathrm{i}$; (2) $\eta_1 = \sqrt{\mu_1/\varepsilon_1}$; (3) $\eta_2 = \sqrt{\mu_2/\varepsilon_2}$; (4) for nonmagnetic media, $\eta_2/\eta_1 = n_1/n_2$.			

Total Internal Reflection

$$n_i > n_t$$

External reflection (fast medium): $n_i < n_t \quad and \;\; \theta_t < \theta_i$;

Internal reflection (slow medium): $n_i > n_t \quad and \;\; \theta_t > \theta_i$,
and r_s $is\ positive\ for\ every\ \ \theta_i$

As for **internal reflection**, there is a $\theta_i = \theta_c \;\; (critical\ angle)\ where\ both$ $\boldsymbol{r_s}$ and $\boldsymbol{r_p}$ $tends\ to\ unity\ (+1,\ see\ Figure\ 4.41)$, i.e. the amplitudes of the internal incident and of the internal reflected waves are equal. In other words, the propagating internal wave is fully reflected internally and EMF doesn't leave~~s~~ the incident medium --- a phenomena known as **total internal reflection.**

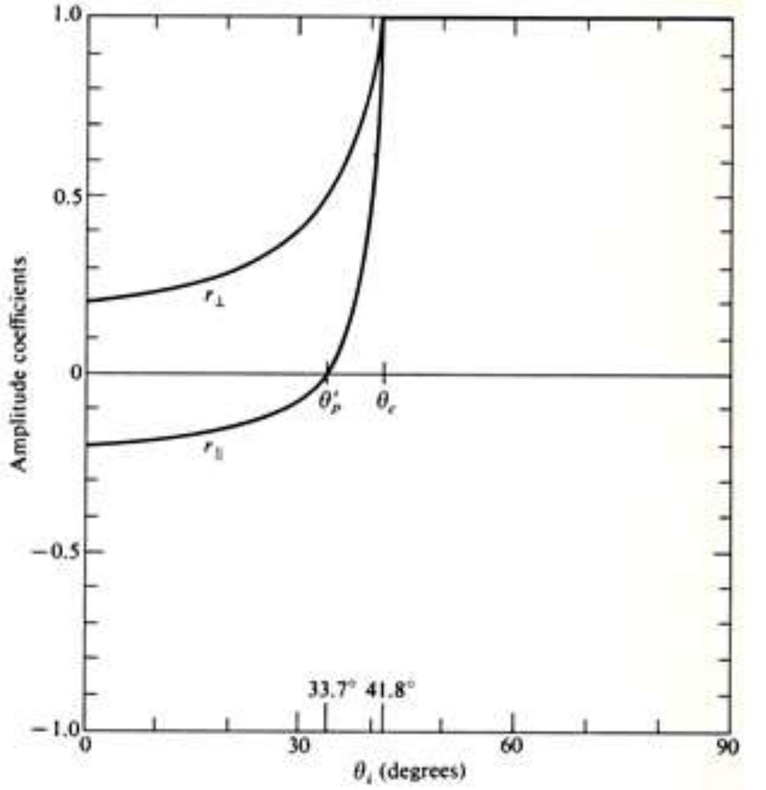


FIGURE 4.41 The amplitude coefficients of reflection as a function of incident angle. These correspond to internal reflection $n_t < n_i$ at an air–glass interface ($n_{ti} = 1/1.5$).

$$\boldsymbol{\theta_c \;\; is\ a\ special\ value\ of\ \theta_i\ for\ which\ \ \theta_t = \frac{\pi}{2}}.$$

Applying $\theta_t = \frac{\pi}{2}$ $into$ Snell' law, where $n_i > n_t$, gives:

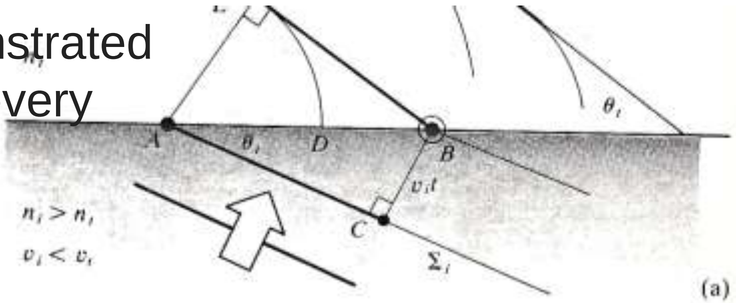
$$\sin\boldsymbol{\theta_c} = \frac{n_t}{n_i} \equiv n_{ti} < 1 \;\; and \;\; \theta_t > \theta_i$$

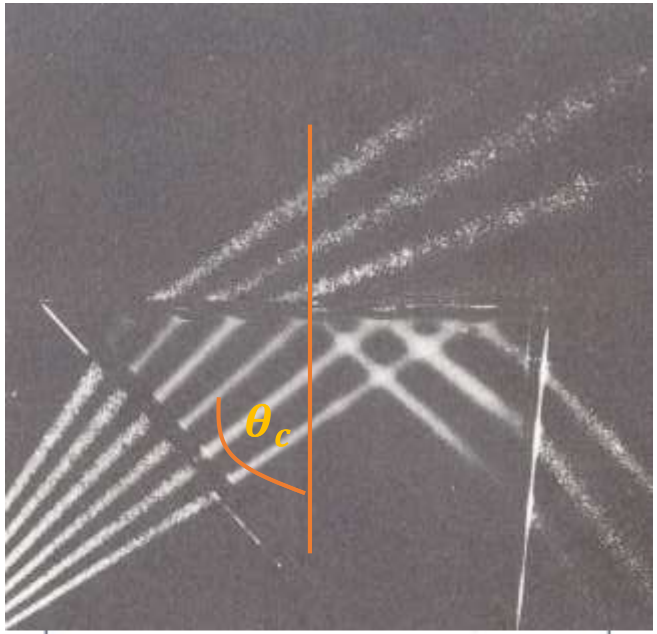
Summary: as $\boldsymbol{\theta_i}$ becomes larger, the transmitted ray gradually approaches tangency with the interface plane (boundary) and consequently more energy appears in the reflected beam.
The larger n_i, the smaller n_{ti} and $\boldsymbol{\theta_c}$ are. For all $\theta_i \geq \theta_c$, all the incoming energy is reflected back into the 'incident medium', a process called **total internal reflection**.

TABLE 4.2 Critical Angles

n_{it}	θ_c (degrees)	θ_c (radians)	n_{it}	θ_c (degrees)	θ_c (radians)
1.30	50.2849	0.8776	1.50	41.8103	0.7297
1.31	49.7612	0.8685	1.51	41.4718	0.7238
1.32	49.2509	0.8596	1.52	41.1395	0.7180
1.33	48.7535	0.8509	1.53	40.8132	0.7123
1.34	48.2682	0.8424	1.54	40.4927	0.7067
1.35	47.7946	0.8342	1.55	40.1778	0.7012
1.36	47.3321	0.8261	1.56	39.8683	0.6958
1.37	46.8803	0.8182	1.57	39.5642	0.6905
1.38	46.4387	0.8105	1.58	39.2652	0.6853
1.39	46.0070	0.8030	1.59	38.9713	0.6802
1.40	45.5847	0.7956	1.60	38.6822	0.6751
1.41	45.1715	0.7884	1.61	38.3978	0.6702
1.42	44.7670	0.7813	1.62	38.1181	0.6653
1.43	44.3709	0.7744	1.63	37.8428	0.6605
1.44	43.9830	0.7676	1.64	37.5719	0.6558
1.45	43.6028	0.7610	1.65	37.3052	0.6511
1.46	43.2302	0.7545	1.66	37.0427	0.6465
1.47	42.8649	0.7481	1.67	36.7842	0.6420
1.48	42.5066	0.7419	1.68	36.5296	0.6376
1.49	42.1552	0.7357	1.69	36.2789	0.6332

The Total internal reflection can be nicely demonstrated via Huygens–Fresnel principle which says that every point on a wavefront is itself the source of spherical wavelets (Fig. 4.55).

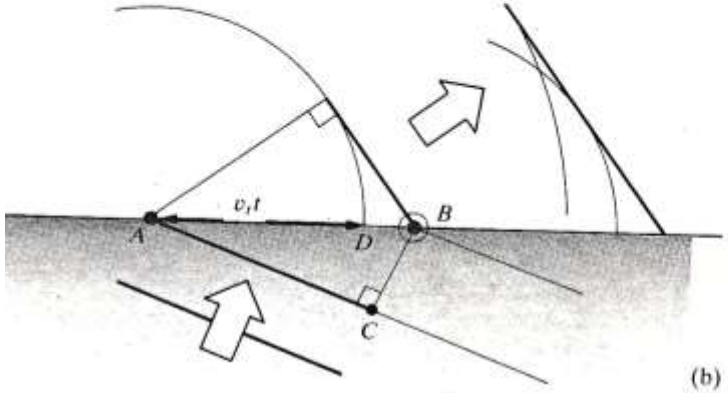




(c) $n_1 > n_2$ and $\theta_i = \theta_c$

Figure 8-10: Snell's laws state that $\theta_r = \theta_i$ and $\sin\theta_t = (n_1/n_2)\sin\theta_i$. Refraction is (a) inward if $n_1 < n_2$ and (b) outward if $n_1 > n_2$; and (c) the refraction angle is 90° if $n_1 > n_2$ and θ_i is equal to or greater than the critical angle $\theta_c = \sin^{-1}(n_2/n_1)$.

In more general terms, when the evanescent wave extends with appreciable amplitude across the rare medium into nearby regions occupied by a higher index material, energy may flow through the ~~gape~~ in what is known as **frustrated total internal reflection** (FTIR).



The evanescent wave, having crossed the ~~gape~~, is still strong enough to drive electrons in the "frustrating" medium. The driven electrons will, in turn, generate a wave that completely alters the original field configuration, permitting energy flow.

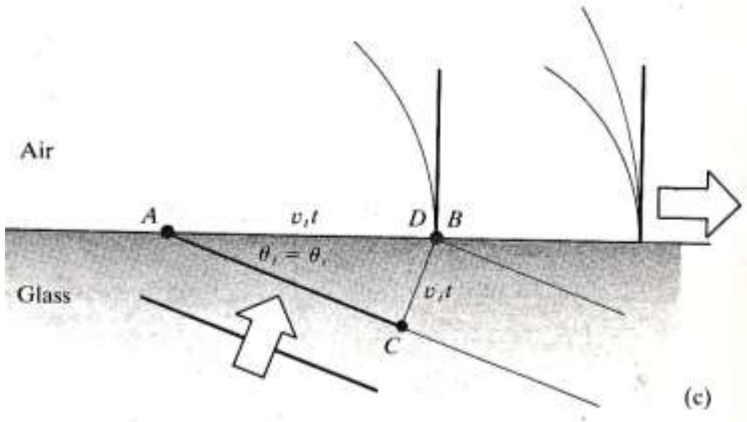


Figure 4.55, Left (Hecht):
θ_i and n_i are kept constant in the sub-figures (a), (b), and (c).
However, from (a) to (c),
n_t decreases thereby increasing v_t.
The reflected beam is not shown.

A glance into the mathematical aspect of Evanescent Wave

The wave function of the transmitted EMF is: $\bar{E}_t = \bar{E}_{0t} \exp i(\bar{k_t} \cdot \bar{r} - \omega t) = \bar{E}_{0t} \exp i(k_{tx} r_x + k_{ty} r_y - \omega t) =$

$$= \bar{E}_{0t} \exp i(k_t sin\theta_t r_x + k_t cos\theta_t r_y - \omega t) \qquad [1]$$

As we are aware that the wave~~s~~ component that travels along the **y direction** at the rare (fast) medium (n_t) is attenuated, we are interested in the content of $cos\theta_t$ of [1]. These can be derived from Snell's equation as follows:

$$n_i sin\theta_i = n_t sin\theta_t = n_t(1 - cos^2\theta_t)^{1/2} \Longrightarrow cos\theta_t = \left(1 - \left(\frac{n_i}{n_t}\right)^2 sin^2\theta_t\right)^{1/2} = \left(1 - \frac{sin^2\theta_t}{{n_{ti}}^2}\right)^{1/2}$$

Replacing in [1] $cos\theta_t$ by $\left(1 - \frac{sin^2\theta_t}{{n_{ti}}^2}\right)^{1/2}$, and recalling that we are interested in the case where $\frac{sin^2\theta_t}{{n_{ti}}^2}$>1, the expression

$k_t cos\theta_t = k_t \left(1 - \frac{sin^2\theta_t}{{n_{ti}}^2}\right)^{1/2}$ becomes complex, i.e. $\longrightarrow = \pm i k_t \left(\frac{sin^2\theta_t}{{n_{ti}}^2} - 1\right)^{\frac{1}{2}} \equiv \pm i\beta.$ Introducing the last expression with the pre-exponent negative sign (the positive is not physical) into [1] and following Snell's, i.e. $\boldsymbol{sin\theta_t} = \frac{\boldsymbol{sin\theta_i}}{\boldsymbol{n_{ti}}}$, a wave function with an attenuated **y-component** is born:

$$\bar{E}_t = \bar{E}_{0t}\, e^{-\beta r_y} e^{i(k_t sin\theta_t r_x - \omega t)} = \bar{E}_{0t}\, e^{-\beta r_y} e^{i\left(k_t r_x \frac{\boldsymbol{sin\theta_i}}{\boldsymbol{n_{ti}}} - \omega t\right)}$$

This wave propagates in the x direction as a surface/evanescent wave. Its amplitude decays rapidly in the y-direction into the rare medium, becoming negligible at a distance into the rare medium of only a **few wavelengths gap**.

Interestingly enough, the phase difference between the incident and reflected waves (at the interface) can be shown to differ from π and hence cannot cancel each other. Consequently, by the continuity of the tangential component of $\bar{E}$, there must exist an oscillatory field in the fast medium which vibrates at frequency ω in parallel to the interface, within the plane of incidence.

Regarding energy conversation at the gap environment (proximity), a more extensive treatment would have shown that energy is actually circulating back and forth across the interface, resulting on the average in a zero net flow through the boundary into the less dense medium. However, the energy associated with the evanescent wave that propagates along the boundary in the plane of incidence, is attributed to the fact that in reality, the incident beam would have a **finite cross section** and hence obviously differ from a **true plane wave** (where the plane is considered an infinite unbound plane). This deviation gives rise (via diffraction) to a slight transmission of energy across the interface, which is manifested in the evanescent wave. (Hecht p. 125)

Reflection from a metal (excellent conductor)

1. Conductor (metal) has free electrons, i.e. being able to circulate within the conductor.
2. Their motion constitutes currents.
3. The deriving electric field $\bar{E}$, the resulting current density $\bar{J}$, and the conductivity are related by: $\bar{J} = \sigma\bar{E}$.
4. Dielectric has no free or conducting charges, i.e. $\sigma = 0$, while for metals $\sigma \neq 0$. However
5. **For idealized "perfect" conductor $\boldsymbol{\sigma} \rightarrow \infty$** and free electrons adequately follow the driven field alterations,
6. In perfect conductor no (a) restoring force, (b) natural frequencies and (c) absorption exist, just **re-emission**.
7. In real metals the conduction electrons undergo collisions with the thermally agitated lattice and/or imperfections through which EM energy is irreversibly converted into Joule heat.

We previously show that for a good conductor $\frac{\sigma}{\varepsilon^R \omega}$>>1 in the visible range, and hence k and n becomes complex:

$$\tilde{k} = \sqrt{\mu\sigma\omega} \cdot i^{1/2} = \sqrt{\mu\sigma\omega} \cdot \sqrt{\left(\mathrm{e}^{\frac{i\pi}{2}}\right)} = \sqrt{\mu\sigma\omega} \cdot \mathrm{e}^{\frac{i\pi}{4}} \quad \Longrightarrow \quad \left|\tilde{k}\right| = \sqrt{\mu\sigma\omega} = \frac{\omega}{v} = \frac{\omega}{\frac{c}{n}} = \frac{n\omega}{c} \Longrightarrow$$

$$n_{con} = \frac{c}{\omega}\sqrt{\mu\sigma\omega} = c\sqrt{\frac{\mu\sigma}{\omega}}$$

So the larger σ the larger n_{con} which decreases as $f\ increases$. Next, going back to the reflectance of a beam which propagates through air (n_i=1) and perpendicularly hit a good conductor surface, i.e. $n_t = n_{con} = \tilde{n} = n_R + in_I$ we have:

$$R = \left(\frac{n_t - 1}{n_t + 1}\right)^2 \to R_{con} = \left(\frac{\tilde{n}_{con,t} - 1}{\tilde{n}_{con,t} + 1}\right)\left(\frac{\tilde{n}_{con,t} - 1}{\tilde{n}_{con,t} + 1}\right)^* = \frac{(n_R - 1)^2 + {n_I}^2}{(n_R + 1)^2 + {n_I}^2}$$

When $\sigma \to 0, n_I \to 0\ and\ R_{con} \to R_{dia.} = \left(\frac{n_R - 1}{n_R + 1}\right)^2\ for\ n_i$=1 (air-dielectric interface). In this case n_R=n_t

However, for $\sigma \to \infty, n_I \gg n_R\ and\ R_{con} \to 1.$

Phase and Group velocities

Phase velocity of a wave is the rate at which its phase (the argument of the periodic function, $\phi = kx - \omega t$) is changed with time, for a single frequency. In other words, it is the velocity at which the phase of any one frequency component of the wave travels. The phase velocity is given in terms of the wavelength λ and period T as $v_p = \frac{\lambda}{T} \to \frac{\omega}{k}$. From $\phi = kx - \omega t \Longrightarrow \frac{\partial \phi}{\partial x} = k$ and $-\frac{\partial \phi}{\partial t} = \omega$, out of which $v = \frac{dx}{dt} = \frac{\omega}{k}$

Group velocity, $\frac{\Delta\omega}{\Delta k}$, is the rate at which the amplitude of the wave package envelope (usually resulted from the wave modulation) changes in time. In general ω *is a function of* k.

Example: By combining two sines with slightly different frequencies and wavelengths (wave #s), one gets:

Envelope wave Carrier wave

$$\cos[(k-\Delta k)x-(\omega-\Delta\omega)t]+\cos[(k+\Delta k)x-(\omega+\Delta\omega)t] = \mathbf{2\cos(\Delta kx - \Delta\omega t)} \cdot \mathbf{\cos(kx-\omega t)}$$

So that the envelope (group) velocity is $\boldsymbol{v_g = \frac{\Delta\omega}{\Delta k} \to \frac{d\omega}{dk}}$ *and* the phase velocity is $\boldsymbol{v_p = \frac{\omega}{k}}$

Recalling that $n = \frac{c}{\boldsymbol{v_p}} = \frac{ck}{\boldsymbol{\omega}} \Longrightarrow \omega = \frac{ck(n)}{\boldsymbol{n}} \Longrightarrow v_g = \frac{d\omega}{dk} = \frac{c}{n} - \frac{ck}{n^2} \cdot \frac{dn}{dk}$

where $\frac{c}{n} = v_p$ and equals v_g *only for* $\frac{dn}{dk} = 0$, a circumstance where the phase and group velocities are equal.

The three macroscopic properties: ε, μ and σ which characterize medium are actually, in different extents, frequency (of the perturbation EMF) dependent.

Obviously, the frequency dependence of the permittivity makes n, the refractive index, and v, the wave propagation velocity, frequency dependent as well, a phenomenon called **dispersion**.

Next, because waves of different frequency travel~~s~~ at different speeds in a dispersive medium, a wave form that incorporates a range of frequencies will change shape (being modulated) as it propagates. For instance, a sharply defined sinusoidal wave will typically flatten~~s~~ out, whereas each sinusoidal component travels at the ordinary **wave** (or **phase**) **velocity**: $\boldsymbol{v_p = \frac{\omega}{k}}$, the **packet as a whole (**the **"envelope")** travels at the so-called **group velocity:** $\boldsymbol{v_g = \frac{d\omega}{dk}}$

The energy carried by a wave packet in a dispersive medium ordinarily travels at the ***group*** velocity, not the phase velocity.

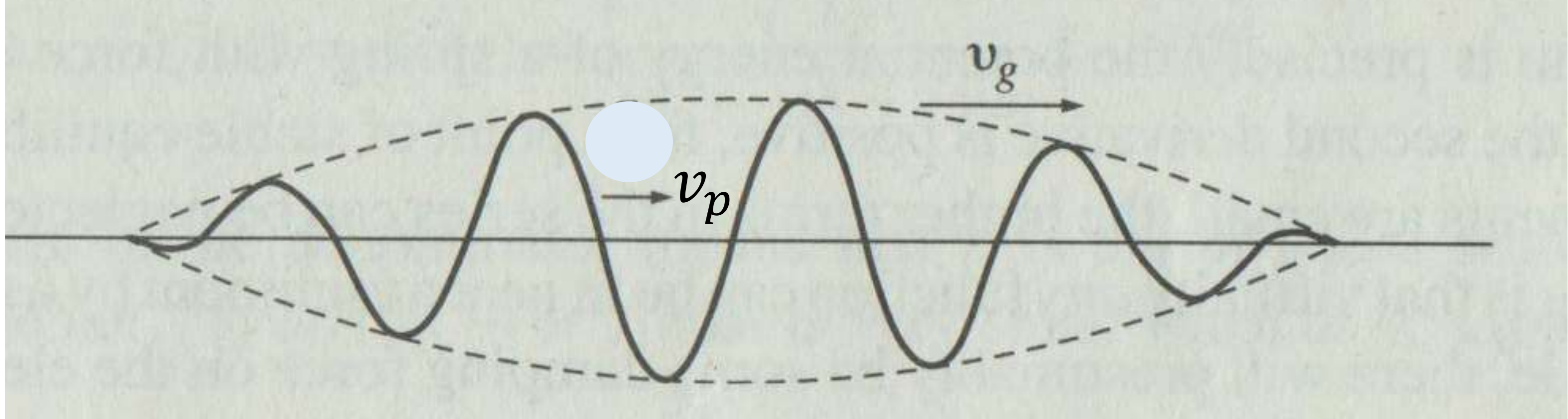


Figure 9.20 (Griffiths, p399)