Chapter 5:

Metallic waveguides and cavities

- Until now we have discussed wave solution of Maxwell's equations with plane wave fronts (i.e. having the same phase), unlimited in extent (unbound).
- This resulted in a transverse EM (TEM) wave having an infinite unbound wave front.
- On the other hand, we are interested in **transmission, along a chosen direction**, of EM power which can be realized lengthwise **through a hollow line in a conductor** (see Figure 1).
- When this happens, the propagating wave is **confined to the interior** of the hollow space.

From Figure 1:

What are the wave types which can propagate along z direction in the hollow space?

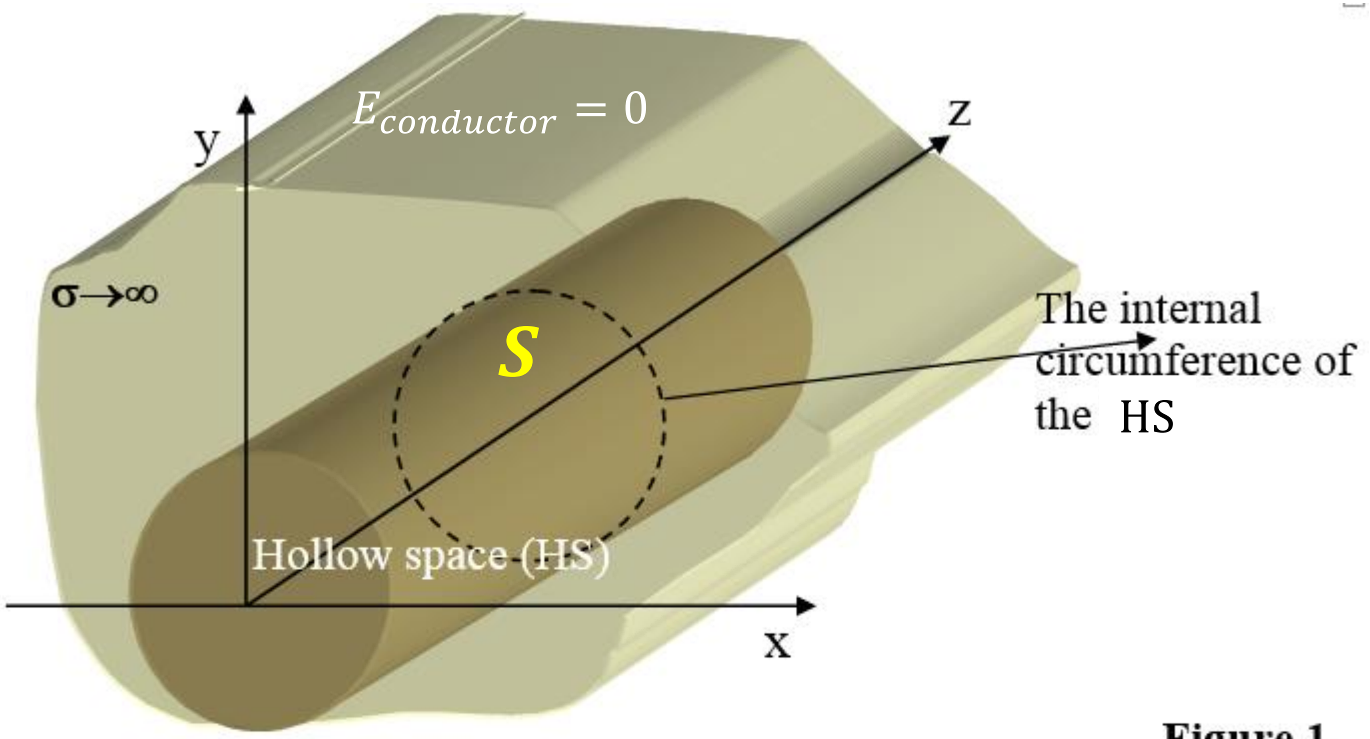
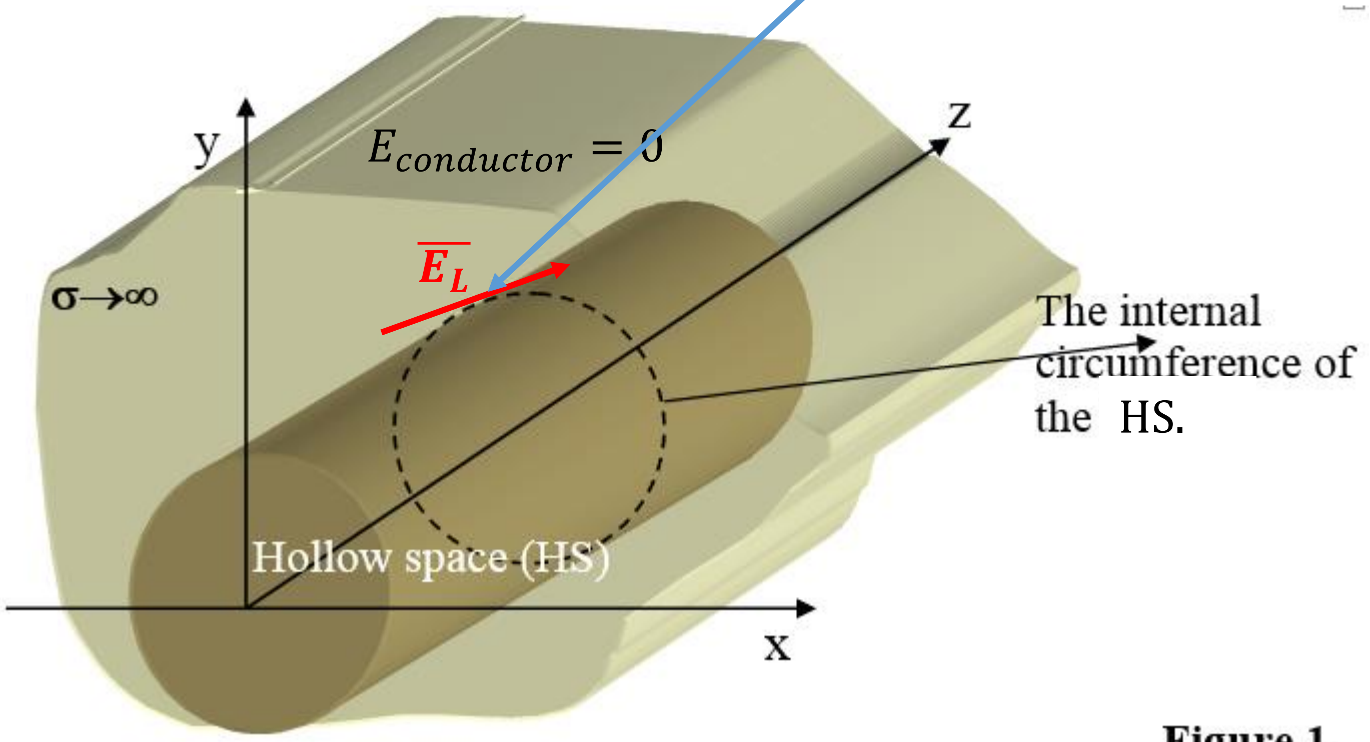


Figure 1

1. Plane wavefront **TEM (which we learn until now).** As propagated along the z direction, then $E_z = 0,\ B_z = 0$ and the field components at the x-y plane (for instance the plane defined by the dashed circle) must be indifferent of x and/or y, in other words $\frac{\partial}{\partial x}$ **and** $\frac{\partial}{\partial y}$ must equal **zero.**

   Therefore , under the continuity boundary conditions, i.e. $E_L = 0$, and being the wavefront **planar, then** the value of the fields $E_x\ and/or\ E_y$ within this plane, must equal that on the circumference (dashed line), i.e. $= 0$.
   So we end with a confined plane wave whose $E_x = E_y = E_z = 0$ and hence we conclude that:



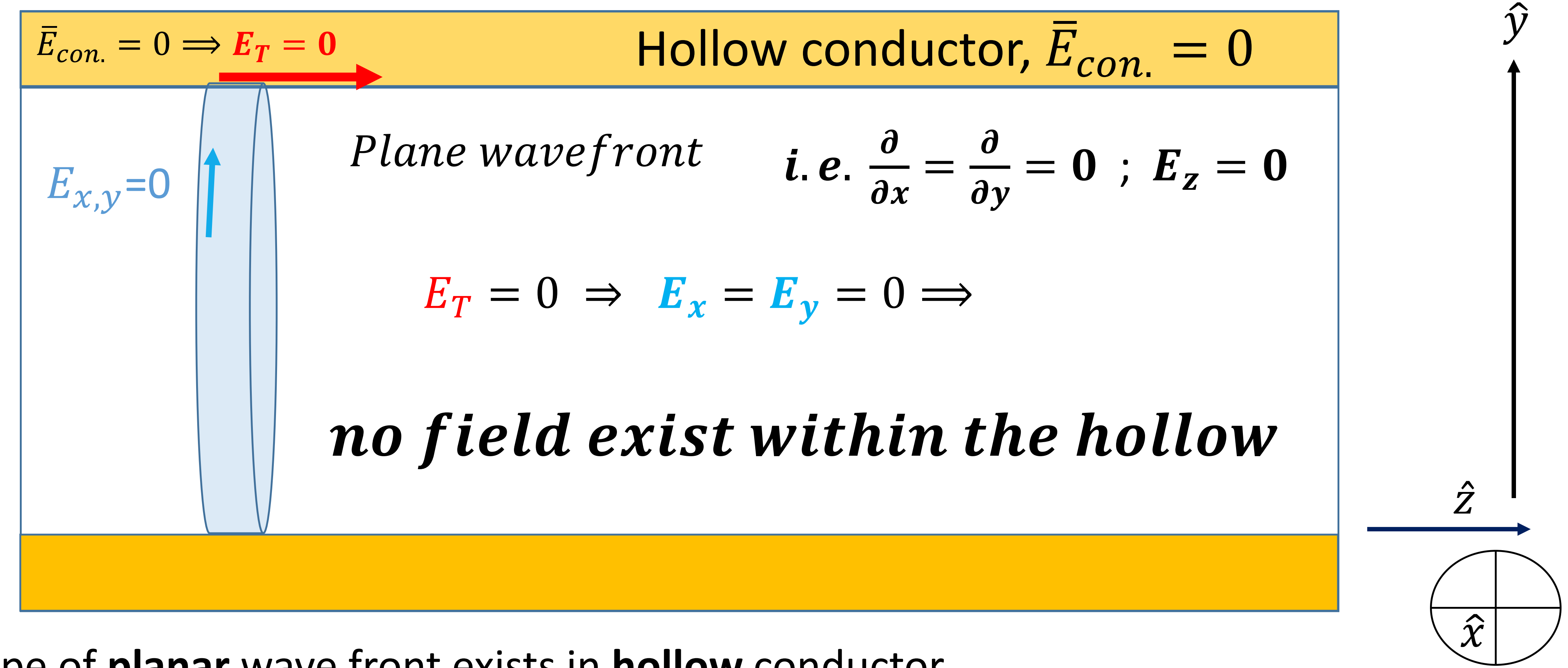
A plane TEM wave with wavefront perpendicular to guide axis cannot be propagated, i.e. Its energy cannot be transmitted through a conducting hollow space.

Figure 1

2. **Can TEM wave which propagates along z direction $(E_z = 0, B_z = 0)$ and has no plane wavefront, exist?** i.e.

- $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} \neq 0$
- From TEM $\Longrightarrow \boldsymbol{E_z = 0, B_z = 0} \Longrightarrow \frac{\partial E_z}{\partial t} = \frac{\partial B_z}{\partial t} = 0$ *and hence according to Maxwell's equations III and IV*:
- The *z components of* $\boldsymbol{curl\ \bar{E}}$ and $\boldsymbol{curl\ \bar{B}}$ are zero as well.
- Thus, the electric field component within the $\mathrm{x - y}$ plane, $\bar{E}_{xy}$, must be a conservative vector (electric) field, hence
- *satisfying*: $\bar{E}_x = -\frac{\partial V}{\partial x}$ *and* $\bar{E}_y = -\frac{\partial V}{\partial y}$. Furthermore, since in the present case $div\bar{E} = 0 = \rho$
- **(divgrad)** $\boldsymbol{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0}$
- Going back to Figure 1 and specifically to the inner circumference marked by dashed line, the boundary conditions dictates: $-\frac{\partial V}{\partial L} = \bar{E}_T = \bar{E}_{\boldsymbol{L}} = 0$ *and hence*:
- $V = constant\ all\ over\ the\ conducting\ inner\ surface\ of\ the\ hollow$
- Hence these two characteristics of V: $\boldsymbol{del^2_{x,y} = 0\ and\ V_L = constant}\ defines\ V\ as\ an\ \boldsymbol{elecrostatic\ potential}$.
- Hence, the electric field within the hollow in every $x - y\ plane$ must be similar to that which exists in charged conductor whose surface corresponds to that of the hollow space. Next,
- Since within a conductor $E = 0$, *even when charged*, (*unless there is an inner hollow* –

a. **Can** $TEM\ (\boldsymbol{E_z = 0})$ *wave with* ***planar*** *wave front exist* within a **hollow** conductor?



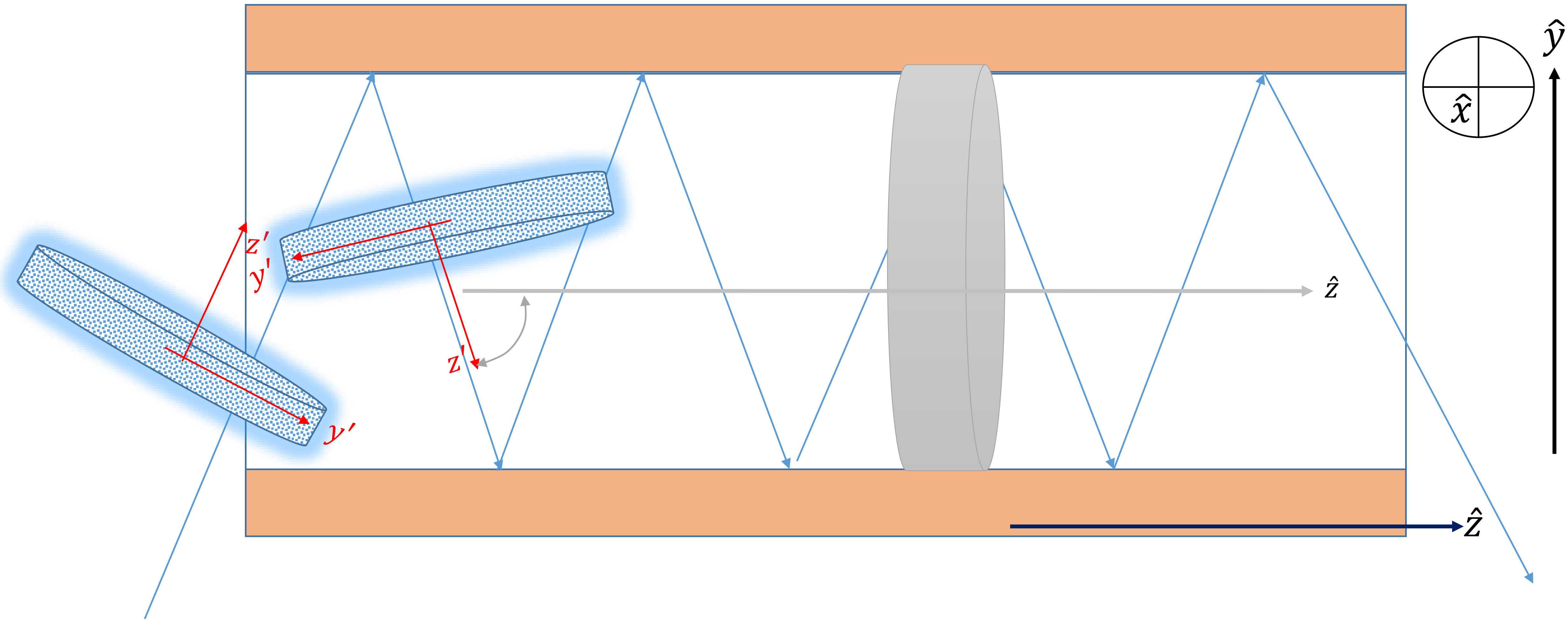
- No **TEM** type of **planar** wave front exists in **hollow** conductor.
- Since $E_x = E_y = 0$ and *given that* $E_z = 0 \Longrightarrow Curl\bar{E} = 0 = -\dot{\bar{B}} \Longrightarrow \bar{B} = \boldsymbol{constant, zero}$ $\Longrightarrow$ *No EM wave with planar wave front can propagate along* $\hat{z}$ within the **hollow** conductor.

So in summary, it is concluded that the propagation of EMF wave of TEM type along a hollow space within a conductor is impossible.

Propagation of EM wave ***oblique*** to the axis of the hollow space in conductor.

- $\bar{k}_{particular} \nparallel \hat{z} \Longrightarrow multiple\ reflection\ of\ (the\ particular\ wave)$ $from\ the\ interior\ metallic\ walls, which$
- interfere with each other to form standing wave within the $x-\mathrm{y}$ ($\boldsymbol{S}-Fig.2$) plane, $\perp \hat{z}$, the hollow's cross section.
- Hence, though the particular wave is oblique to $\hat{z}$, and its carried fields are $\bar{E} \perp \bar{B} \perp \bar{k}_{particular}$, there will be always a field component in the direction of $\hat{z}$, i.e parallel to the hollow's axis.
- Two principal EM waves exist when propagating ***oblique*** to the axis of the hollow space in conductor:
- **Transverse Electric ($\boldsymbol{TE},\ \boldsymbol{B-Wave}$) and transverse Magnetic ($\boldsymbol{TM},\ \boldsymbol{E-Wave}$)**.

Propagation of EM wave ***oblique*** - an illustration



$$TM; \quad E - wave:$$

$$B_z = 0, E_z \neq 0 \Longrightarrow \text{a longitudinal wave of } E_z.$$

$$TE; \quad B - wave:$$

$$E_z = 0, B_z \neq 0 \Longrightarrow \text{a longitudinal wave of } B_z.$$

Both TM and TE waves **have no planar wavefront**

Another solution can be a combination of TM and TE waves. However, their propagation along the hollow's axis is **not TEM** and **has no planar wavefront**

Modes and frequency range of the standing waves in $x - \mathrm{y}$ plane, $\perp$ $to\ \hat{z}$, are dependent on the hollow's cross-section dimensions and the frequency of the carrier wave. Since along the z direction there are no restrictions, the EM disturbance spreads along it as a free wave, having a typical wave # k_{guide}.

General: The mathematical expression of the two **longitudinal wave components** are:

$$\text{For}\quad \textbf{TM}\ (B_z = 0):\ E_{z0}\,\hat{z}e^{i(k_g z-\omega t)}\hat{z} \quad \text{and for} \quad \textbf{TE}\ (E_z = 0):\ B_{z0}\,e^{i(k_g z-\omega t)}\,\hat{z} \qquad [1]$$

Note:

- E_{z0} and B_{z0} are not constant as they were on plane wavefront, but vary across the x-y plane and thus are y and x dependent.
- This dependency will be found via Maxwell's equation under the mechanical restriction of the guide, the fact that $\sigma \to \infty$ *and* the boundary continuity $\Delta E_{tangent} = 0$.
- The rectangular hollow space dictates the use of **Cartesian coordinates** and hence from [1] the following conversions are yielded:

$$\frac{\partial}{\partial z} \to ik_g \qquad and \qquad \frac{\partial}{\partial t} \to -i\omega \qquad [2]$$

- As a result: $\text{Curl} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_y \end{vmatrix} \to \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & ik_g \\ E_x & E_y & E_y \end{vmatrix}$ and since $\omega = k_0 c$, $\frac{\partial}{\partial t} \to i\omega = ik_0 c$

Equations a-d represent a spectrum of solutions for the fields components in the x-y planes (hollow space cross sections), out of which the expression for E- and B-waves in the hollow space is determined as follows.

Extraction of E_x:

Multiplying $Eq. II$ by k_g and $Eq. III$ by $-k_0 c$ even the coefficients of B_y in the two equations. Adding them and rearranging the equation gives:

$$E_x = \frac{i}{(k_0^2 - k_g^2)}\left[k_g \frac{\partial E_z}{\partial x} + c k_0 \frac{\partial B_z}{\partial y}\right] \qquad [a]$$

Extraction of B_y:

Similarly, multiplying $Eq. II$ by ik_0 and $Eq. III$ by $-ick_g$ equals the coefficients of B_y in the two equations. Adding the them and rearranging the equation gives:

$$B_y = \frac{i}{(k_0^2 - k_g^2)}\left[\frac{k_0}{c}\frac{\partial E_z}{\partial x} + k_g \frac{\partial B_z}{\partial y}\right] \qquad [b]$$

Similar treatment of $Eqs. I\ and\ IV\ gives$:

$$B_x = \frac{i}{(k_0^2 - k_g^2)}\left[-\frac{k_0}{c}\frac{\partial E_z}{\partial y} + k_g \frac{\partial B_z}{\partial z}\right] \qquad [c]$$

and

$$E_y = \frac{i}{(k_0^2 - k_g^2)}\left[k_g \frac{\partial E_z}{\partial y} - c k_0 \frac{\partial B_z}{\partial x}\right] \qquad [d]$$

TM (E-wave)

Here we just need to substitute $B_z = 0$ in equations a-d. This left us with the following expressions:

$$E_x = \frac{i}{(k_0^2 - k_g^2)} k_g \frac{\partial E_z}{\partial x} \quad ; \quad E_y = \frac{i}{(k_0^2 - k_g^2)} k_g \frac{\partial E_z}{\partial y} \qquad [e]$$

$$B_x = -\frac{i\, k_0/c}{(k_0^2 - k_g^2)} \frac{\partial E_z}{\partial y} \quad ; \quad B_y = -\frac{i\, k_0/c}{(k_0^2 - k_g^2)} \frac{\partial E_z}{\partial x} \qquad [f]$$

Next, from $[e]$ we get:

$$\frac{E_x}{E_y} = \frac{\frac{\partial E_z}{\partial x}}{\frac{\partial E_z}{\partial y}} \rightarrow extractibng\ these\ derivations\ from\ [f] \Longrightarrow \frac{E_x}{E_y} = -\frac{B_y}{B_x} \Longrightarrow E_x = -E_y \frac{B_y}{B_x} \Longrightarrow \qquad [g]$$

$$\Longrightarrow (\bar{E} \cdot \bar{B})_{x-y\ plane} = \left(-E_y \frac{B_y}{B_x} \hat{x} + E_y\, \hat{y}\right) \cdot \left(\bar{B}_x \hat{x} + \bar{B}_y \hat{y}\right) = 0$$

Conclusion: within the cross section of the hollow space $(x - y\ plane)$: $\bar{E} \perp \bar{B} \perp k_g$ and $\bar{E}, \bar{B}$ are in phase.

Similar treatment yields similar results for B-wave, i.e. TE $(E_z = 0)$ wave.

And what about the longitudinal component $\boldsymbol{E_z(r,t)}$ of a TM wave? All that can be said is that it obeys the differential wave equation:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} \qquad [h]$$

Next, Introducing $E_z = E_{0,z}e^{i(k_g z - \omega t)}$ into $[h]$ $yields$:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = \left({k_g}^2 - \frac{\omega^2}{c^2}\right)E_z = \left({k_g}^2 - {k_0}^2\right)E_z \qquad [i]$$

Separation of variables is employed to solve (the amplitude) $E_{0,z}$, i.e. $by\ introducing\ E_{0,z} = X(x)Y(y)\ into\ [i]$:

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = \left({k_g}^2 - {k_0}^2\right)XY \qquad [j]$$

$and\ d$ividing [j] by $XY\ gives$: $\frac{X''}{X} + \frac{Y''}{Y} = \left({k_g}^2 - {k_0}^2\right)$ $\qquad [k]$

Since $[k]$ holds true for every $x\ and\ y$ $\quad \Longrightarrow \quad \frac{X''}{X} = -p^2 \quad and \quad \frac{Y''}{Y} = -q^2.$

$p\ and\ q\ are\ integers$. Substituting these relations into $[k]$ gives: $p^2 + q^2 = {k_0}^2 - {k_g}^2$ $\qquad [l]$

The solutions of $X(x)$ *and* $Y(y)$ could be any periodic function. We choose:

$$\boldsymbol{Y(y) = Y_0 sinqy} \qquad \textbf{and} \qquad \boldsymbol{X(x) = X_0 sinpx} \qquad q, p \ are \ integers \qquad [m]$$

The solution of $[m]$ depends on boundary value. In the case of metallic wave guide, in z direction, the tangential (to the inner surface) component, i.e. $E_{tan,\, z} = 0.$ *This is to say that* applying boundary conditions, i.e. $E_z = 0 \ at \ x, \ y = 0 \ \ and \ at \ x = a \ and \ at \ y = b \ \ into \ \ [m]$ gives:

$$X(x = a) = X_0 \sin(pa) = 0 \Rightarrow pa = l\pi \Rightarrow p = \frac{l\pi}{a} \qquad \Rightarrow X(x) = X_0 \sin\left(\frac{l\pi}{a} x\right) \qquad (V)$$

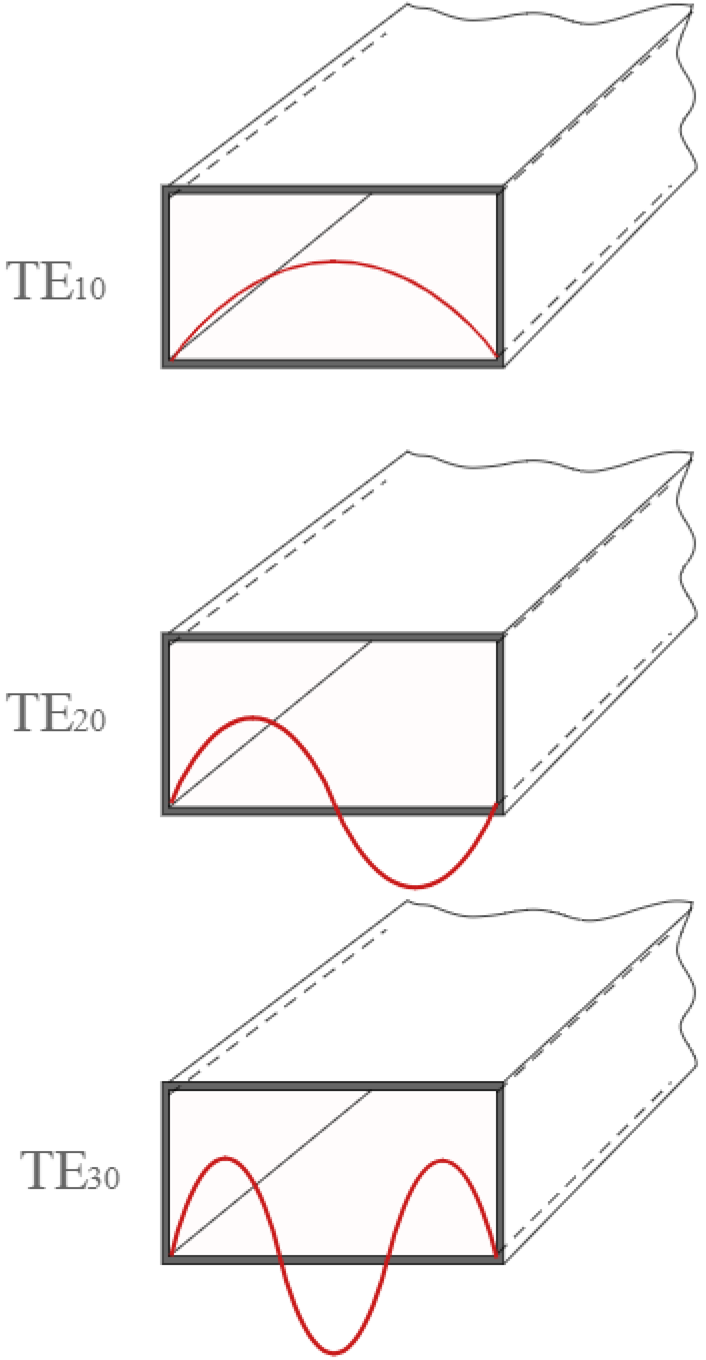
$$Y(y = b) = Y_0 \sin(qb) = 0 \Rightarrow qb = m\pi \Rightarrow q = \frac{m\pi}{b} \qquad \Rightarrow Y(y) = Y_0 \sin\left(\frac{m\pi}{b} y\right) \qquad (VI) \quad l, m \ are \ integers$$

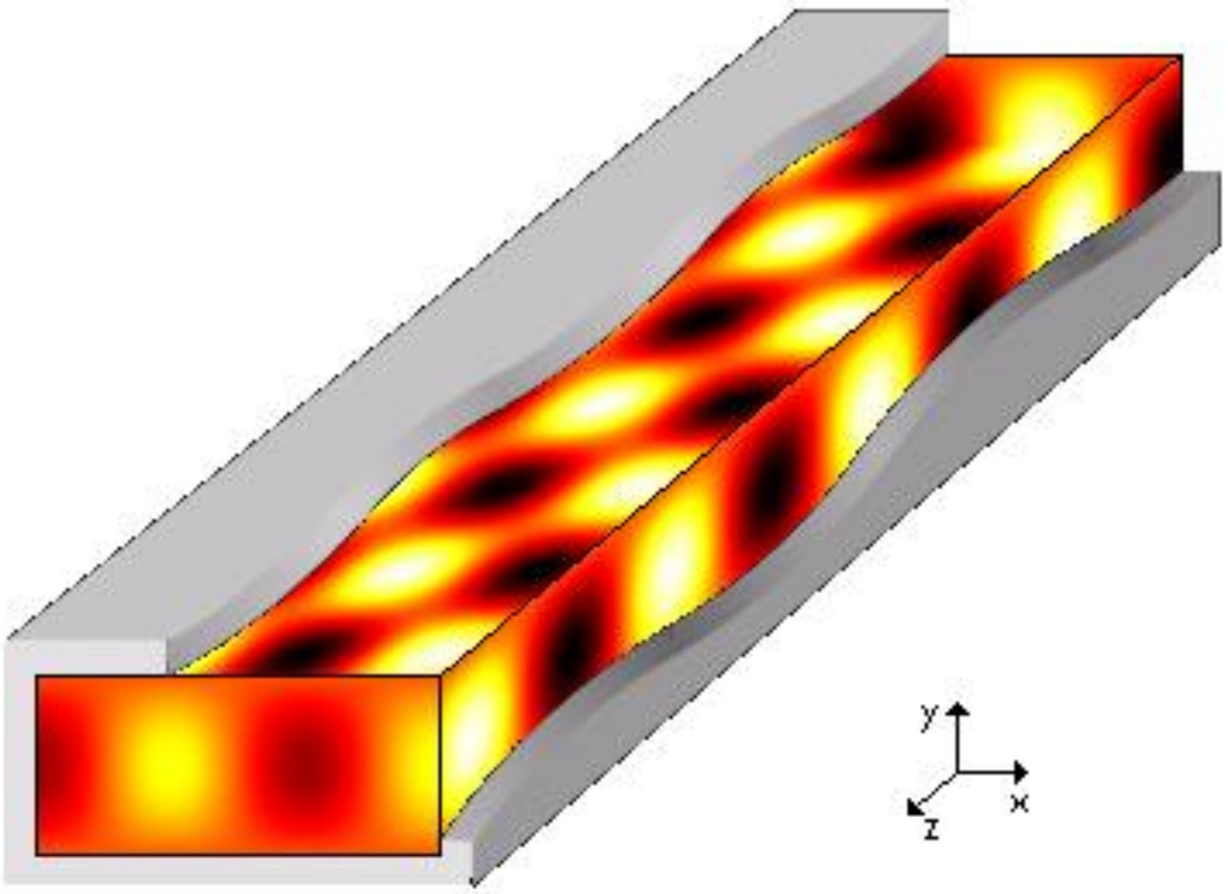
Replacing the amplitude of $\boldsymbol{E_{z0}}\ \hat{z} e^{i(k_g z - \omega t)} \hat{z}$ by the multiplication $E_{0,z} = X(x)Y(y), i.e. (\mathrm{V}) \cdot (VI)$, yields the complete solution of the **longitudinal** component:

$$\bar{E}_z = E_0 \sin\left(\frac{\pi l x}{a}\right) \sin\left(\frac{\pi m y}{b}\right) e^{i(k_g z - \omega t)} \hat{z} \qquad (VII)$$

Envelope wave — Carrier wave

This is a standing wave within the cross section of the hollow space, i.e. x-y plane. Its amplitude is time independent. The Carrier wave is amplitude modulated. The envelope standing wave has antinodes and nodes.


TE_{10}
TE_{20}
TE_{30}



Electric field Ex component of the TE31 mode inside an x-band hollow metal waveguide.

Example of waveguides in an air traffic control radar

Waveguide supplying power for the Argonne National Laboratory Advanced Photon Source.

Next, introducing the values $p = \frac{l\pi}{a}$ and $q = \frac{m\pi}{b}$ into $(l), i.e.\ into\ \ p^2 + q^2 = {k_0}^2 - {k_g}^2$ gives:

$$\left(\frac{l\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 = {k_0}^2 - {k_g}^2 \Longrightarrow \left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2 = \frac{{k_0}^2 - {k_g}^2}{\pi^2} \qquad (VIII)$$

The integers $l\ and\ m$ define the mode of the standing wave $in\ a\ given\ x - y\ plane$, its **nodes** and **antinodes.** Those waves are indicated by the indexes $\boldsymbol{l\ and\ m}\ as\ follows$: $TM_{lm}\ and\ TE_{lm}$ of the standing wave E_z. The relation between the **transverse** components E_x, E_y, B_x, B_y and the **longitudinal** component E_z are given by the equations $(e)\ and\ (f)\ above$.

Do the modes TM_{00}, TM_{01} and TM_{10} exist in the hollow space of the wave guide? Yes, No, Why?

Where λ_0 is the carrier (particular wave, that reflected from the inner walls) wave's wavelength in vacuum. The smaller λ_0 the larger λ_g? From IX we conclude that when $\lambda_0 \rightarrow \lambda_{cutoff} \Longrightarrow \lambda_g \longrightarrow \infty, and$ the practical meaning $of\ that\ is\ that\ no\ wave\ is\ propagating\ along\ z\ direction\ along\ the\ waveguide.$ $Therfore, there\ is\ a\ frequency\ f_{cutoff}\ after\ which\ no\ EM\ wave\ exist\ in\ metallic\ waveguide.$

$$\frac{1}{{\lambda_g}^2} = \frac{1}{{\lambda_0}^2} - \frac{1}{4}\left[\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right] \equiv \frac{1}{{\lambda_0}^2} - \frac{1}{{\lambda_{cutoff}}^2} \qquad (IX)$$

Hence, it is concluded that:

The cutoff frequency is dependent on the waveguide dimensions. Waveguide with a given dimension acts as a filter, attenuating waves with frequencies near the cutoff frequency. A non-monochromatic wave undergoes dispersion in a waveguide.

Converting wavelengths to frequencies in Eq. IX, i.e. $\frac{1}{{\lambda_0}^2} \to \frac{f^2}{c^2}$

$$\frac{1}{{\lambda_g}^2} = \frac{1}{{\lambda_0}^2} - \frac{1}{4}\left[\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right] = \frac{f^2}{c^2} - \frac{1}{\lambda_{cutoff}^2} \qquad (X)$$

Furthermore:

Multiplying [X] by c^2 gives:

$$f_{cutoff}^2 = \frac{c^2}{4}\left[\left(\frac{l}{a}\right)^2 + \left(\frac{m}{b}\right)^2\right] \qquad (XI)$$

Example: What is the $\lambda_{cutoff}\ of\ a\ TM_{11}$ wave which travels along a waveguide having a cross section of 3X4 cm?

Introducing the given numbers in Eq. (XI) gives:

$$\frac{1}{{\lambda_c}^2} = \frac{1}{4}\left(\frac{1}{3^2} + \frac{1}{4^2}\right)cm^{-2} = \frac{1}{4}\left(\frac{1}{9} + \frac{1}{16}\right)cm^{-2} = \frac{1}{4}\left(\frac{16+9}{16 \cdot 9}\right)cm^{-2} = \frac{1}{4} \cdot \frac{25}{144}cm^{-2} \qquad \Longrightarrow \lambda_c = 5cm$$