Transmission lines

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Transmission lines are used (as are waveguides) to guide electromagnetic waves from one place to another. A coaxial cable (used, for example, to connect a radio or television to an aerial) is an example of a transmission line. Transmission lines may be less bulky and less expensive than waveguides; but they generally have higher losses, so are more appropriate for carrying low-power signals over short distances.

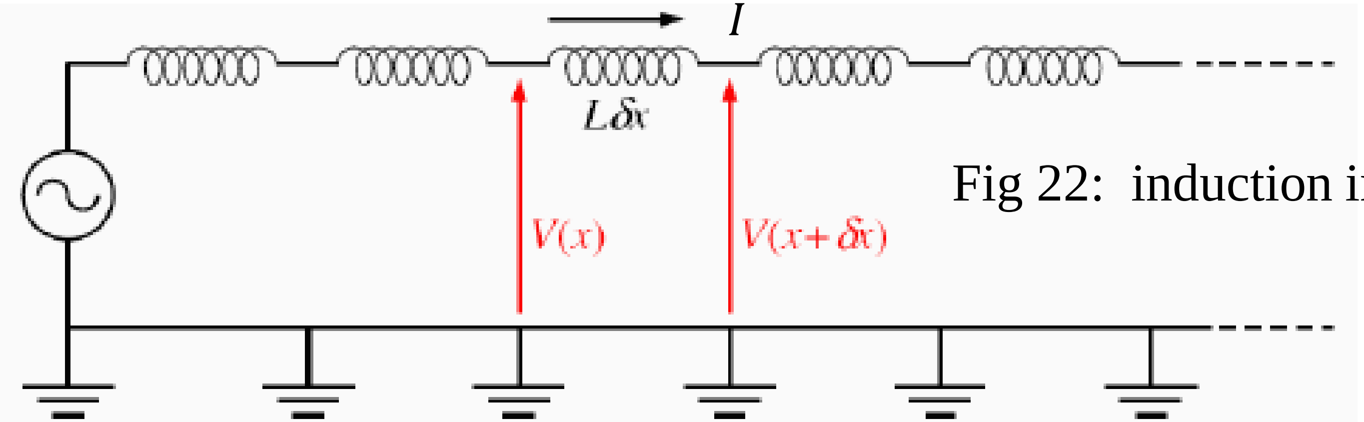


Fig 22: induction in a transmission line

11.1 *LC* model of a transmission line

Consider an infinitely long, parallel wire with zero resistance. In general, the wire will have some inductance per unit length, L, which means that when an alternating current I flows in the wire, there will be a potential difference between different points along the wire (Fig. 22). If V is the potential at some point along the wire with respect to earth, then the potential difference between two points along the wire is given by:

$$\Delta V = \frac{\partial V}{\partial x}\delta x = -L\delta x \frac{\partial I}{\partial t}. \tag{269}$$

$$\frac{\partial V}{\partial x}\cancel{\delta x} = -L\cancel{\delta x}\frac{\partial I}{\partial t}. \qquad (Ld\dot{I} = dV_{emf}) \tag{269}$$

In general, as well as the inductance, there will also be some capacitance per unit length, C, between the wire and earth (Fig. 23). This means that the current in the wire can vary with position:

$$\frac{\partial I}{\partial x}\cancel{\delta x} = -C\cancel{\delta x}\frac{\partial V}{\partial t}. \qquad (CdV = dq) \tag{270}$$

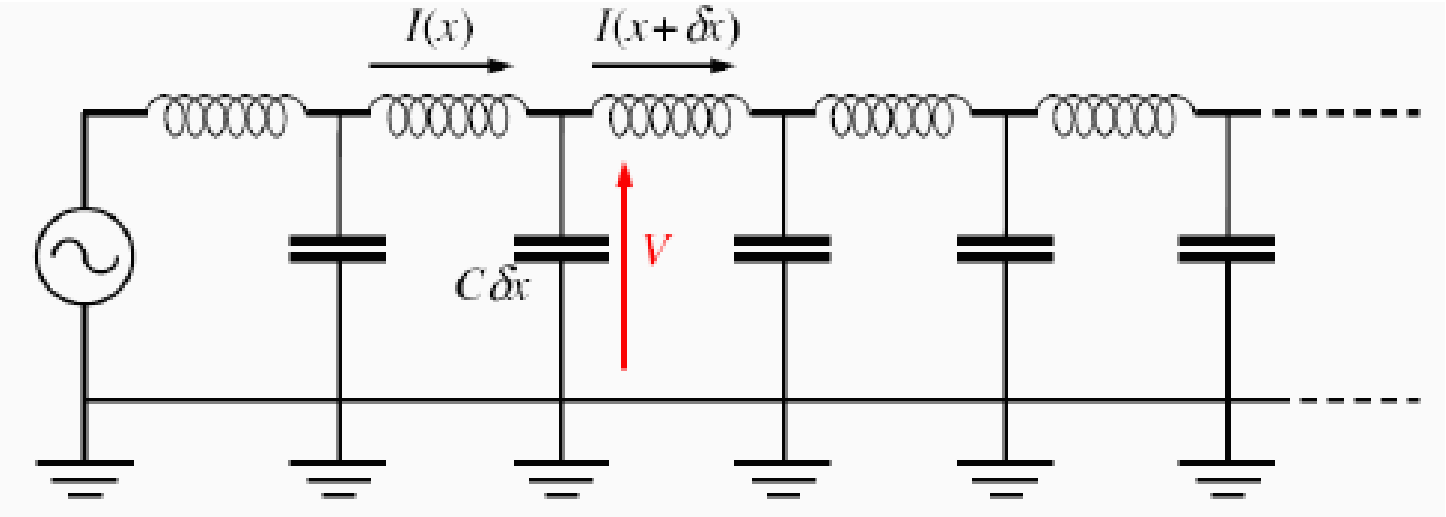


Fig. 23: Capacitance in a transmission line.

From Eqs. 269 and 270 one gets:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}, \tag{271}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \tag{272}$$

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}, \quad (271)$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \quad (272)$$

Differentiating Eq.271 with respect to *t* and of Eq. 272 in respect to *x* yield correspondingly:

$$\frac{\partial^2 V}{\partial x \partial t} = -L\frac{\partial^2 I}{\partial t^2}, \quad (273)$$

$$\frac{\partial^2 I}{\partial x^2} = -C\frac{\partial^2 V}{\partial t \partial x}. \quad (274)$$

Hence:

$$\frac{\partial^2 I}{\partial x^2} = LC\frac{\partial^2 I}{\partial t^2}. \quad (275)$$

Similarly (by differentiating (271) with respect to x and (272) with respect to t), we find:

$$\frac{\partial^2 V}{\partial x^2} = LC\frac{\partial^2 V}{\partial t^2}. \quad (276)$$

Equations (275) and (276) are wave equations for the current in the wire, and the voltage between the wire and earth. The waves travel with speed v, given by:

$$v = \frac{1}{\sqrt{LC}}. \quad (277)$$

The solutions to the wave equations may be written:

$$V = V_0 e^{i(kx-\omega t)}, \quad (278)$$
$$I = I_0 e^{i(kx-\omega t)}, \quad (279)$$

271
$$\boxed{\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t},}$$
272
$$\boxed{\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}.}$$

where the phase velocity is:

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{LC}}. \quad (280)$$

Note that the inductance per unit length L and the capacitance per unit length C are real and positive.

Therefore, if the frequency ω is real, the wave number k will also be real: this implies that waves propagate along the transmission line with constant amplitude. This result is expected, given our assumption about the line having zero resistance.

The solutions must also satisfy the first-order equations 271 and 272 . Substituting the above solutions into these equations, we find:

$$kV_0 = \omega L I_0, \quad (281)$$
$$kI_0 = \omega C V_0. \quad (282)$$

Hence:

$$\frac{V_0}{I_0} = \sqrt{\frac{L}{C}} = Z. \quad (283)$$

Z is the characteristic impedance of the transmission line and measured in ohms Ω .

. Note that, since L and C are real and positive, the impedance is a real number: this means that the voltage and current are in phase. The characteristic impedance of a transmission line is analogous to the impedance of a medium for electromagnetic waves: the impedance of a transmission line gives the ratio of the voltage amplitude to the current amplitude; the impedance of a medium for electromagnetic waves gives the ratio of the electric field amplitude to the magnetic field amplitude.

11.2 Impedance matching

So far, we have assumed that the transmission line has infinite length. Obviously, this cannot be achieved in practice. We can terminate the transmission line using a "load" with impedance Z_L that dissipates the energy in the wave while maintaining the same ratio of voltage to current as exists all along the transmission line – see Fig. 24. In that case, our above analysis for the infinite line will remain valid for the finite line, and we say that the impedances of the line and the load are properly *matched*.

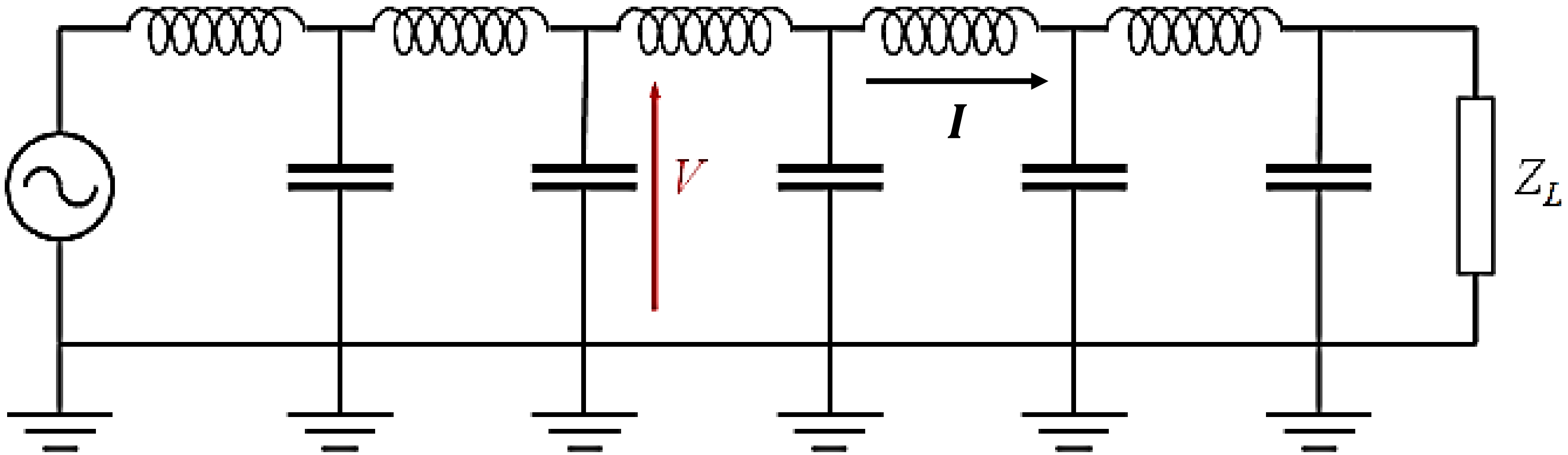


Fig 24: Termination of transmission line with impedance R.

What happens if the impedance of the load, Z_L, is not properly matched to the characteristic impedance of the transmission line, Z?

In that case, we need to consider a solution consisting of a superposition of waves travelling in opposite directions:

$$V = V_0 e^{i(kx-\omega t)} + K V_0 e^{i(-kx-\omega t)}. \tag{284}$$

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}, \quad (271)$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \quad (272)$$

The corresponding current is given by:

$$I = \frac{V_0}{Z} e^{i(kx-\omega t)} - K\frac{V_0}{Z} e^{i(-kx-\omega t)}. \tag{285}$$

Note the minus sign in the second term in the expression for the current: this comes from equations (271) and (272). Let us take the end of the transmission line, where the load is located, to be at $x = 0$. At this position, we have: $V = V_0 e^{-i\omega t}(1+K)$ (286) *and* $\frac{V_0}{Z} e^{-i\omega t}(1-K)$ (287)

If the impedance of the load is Z_L then:

$$Z_L = \frac{V}{I} = Z\frac{1+K}{1-K}. \tag{288}$$

Solving the equation for K, which gives the relative amplitude and phase of the "reflected" wave one gets:

$$K = \frac{Z_L/Z - 1}{Z_L/Z + 1} = \frac{Z_L - Z}{Z_L + Z}. \tag{289}$$

Note: when $Z_L = Z$, $\Longrightarrow$ no reflected wave i.e. the termination is matched to the characteristic transmission line impedance.

11.3 "Lossy" transmission lines

So far, we have assumed that the conductors in the transmission line have zero resistance, and are separated by a perfect insulator. Usually, though, the conductors will have finite conductivity; and the insulator will have some finite resistance. To understand the impact that this has, we need to modify our transmission line model to include:

- a resistance per unit length R in series with the inductance;
- a conductance per unit length G in parallel with the capacitance.

The modified transmission line is illustrated in Fig. 25.

The equations for the current and voltage are then:

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t} - RI, \qquad (290)$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t} - GV. \qquad (291)$$

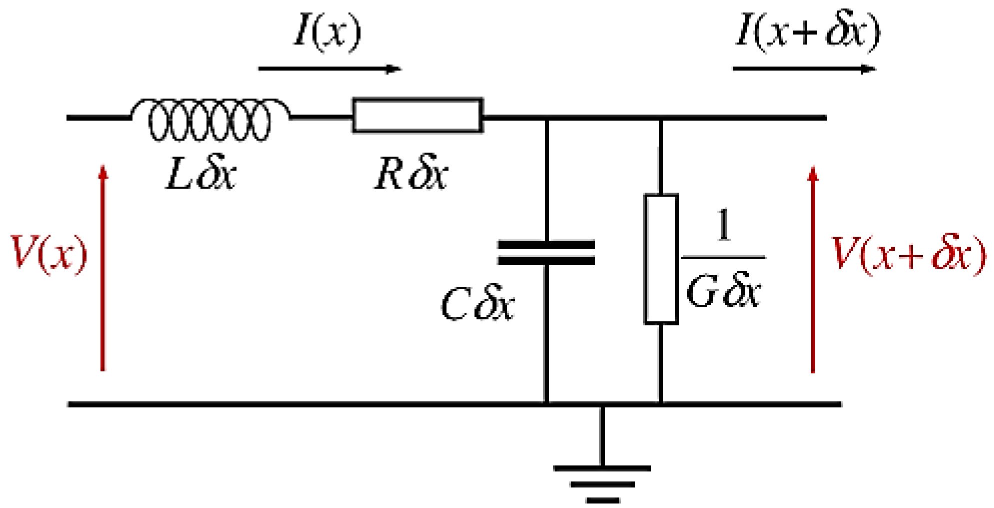


Fig. 25: A "lossy" transmission line.

We can find solutions to the equations (290) and (291) for the voltage and current in the lossy transmission line by considering the case that we propagate a wave with a single, well-defined frequency ω. In that case, we can replace each time derivative by a factor $-i\omega$. The equations become:

$$\frac{\partial V}{\partial x} = i\omega LI - RI = -\tilde{L}\frac{\partial I}{\partial t}, \tag{292}$$

$$\frac{\partial I}{\partial x} = i\omega CV - GV = -\tilde{C}\frac{\partial V}{\partial t}, \tag{293}$$

where

$$\tilde{L} = L - \frac{R}{i\omega} \qquad \text{and} \qquad \tilde{C} = C - \frac{G}{i\omega}. \tag{294}$$

The new equations (292) and (293) for the lossy transmission line look exactly like the original equations (271) and (272) for a lossless transmission line, but with the capacitance C and inductance L replaced by (complex) quantities $\tilde{C}$ and $\tilde{L}$. The imaginary parts of $\tilde{C}$ and $\tilde{L}$ characterise the losses in the lossy transmission line.

$$\frac{\partial V}{\partial x} = -L\frac{\partial I}{\partial t}, \tag{271}$$

$$\frac{\partial I}{\partial x} = -C\frac{\partial V}{\partial t}. \tag{272}$$

Mathematically, we can solve the equations for a lossy transmission line in exactly the same way as we did for the lossless line. In particular, we find for the phase velocity:

$$v = \frac{1}{\sqrt{\tilde{L}\tilde{C}}} = \frac{1}{\sqrt{\left(L + i\frac{R}{\omega}\right)\left(C + i\frac{G}{\omega}\right)}}, \tag{295}$$

and for the impedance:

$$Z = \sqrt{\frac{\tilde{L}}{\tilde{C}}} = \sqrt{\frac{L + i\frac{R}{\omega}}{C + i\frac{G}{\omega}}}. \tag{296}$$

Since the impedance (296) is now a complex number, there will be a phase difference (given by the complex phase of the impedance) between the current and voltage in the transmission line. Note that the phase velocity (295) depends explicitly on the frequency. That means that a lossy transmission line will exhibit dispersion: waves of different frequencies will travel at different speeds, and a the shape of a wave "pulse" composed of different frequencies will change as it travels along the transmission line. Dispersion is one reason why it is important to keep losses in a transmission line as small as possible (for example, by using high-quality materials). The other reason is that in a lossy transmission line, the wave amplitude will attenuate, much like an electromagnetic wave propagating in a conductor.

Recall that we can write the phase velocity:

$$v = \frac{\omega}{k}, \tag{297}$$

where k is the wave number appearing in the solution to the wave equation:

$$V = V_0 e^{i(kx-\omega t)}, \tag{298}$$

and similarly for the current I. Using Eq. (295) for the phase velocity, we have:

$$k = \omega\sqrt{LC}\sqrt{\left(1 + i\frac{R}{\omega L}\right)\left(1 + i\frac{G}{\omega C}\right)}. \tag{299}$$

Let us assume $R \ll \omega L$ (i.e. good conductivity along the transmission line) and $G \ll \omega C$ (i.e. poor conductivity between the lines); then we can make a Taylor series expansion, to find:

$$k \approx \omega\sqrt{LC}\left[1 + \frac{i}{2\omega}\left(\frac{R}{L} + \frac{G}{C}\right)\right]. \tag{300}$$

Finally, we write:

$$k = \alpha + i\beta, \tag{301}$$

and equate real and imaginary parts in equation (300) to give:

$$\alpha \approx \omega\sqrt{LC}, \tag{302}$$

and:

$$\beta \approx \frac{1}{2}\left(\frac{R}{Z_0} + GZ_0\right), \tag{303}$$

where $Z_0 = \sqrt{L/C}$ is the impedance with $R = G = 0$ (not to be confused with the impedance of free space).

Note that since:

$$V = V_0 e^{i(kx-\omega t)} = V_0 e^{-\beta x} e^{i(\alpha x-\omega t)}, \tag{304}$$

the value of α gives the wavelength $\lambda = 1/\alpha$, and the value of β gives the attenuation length $\delta = 1/\beta$.

11.4 Example: a coaxial cable

A lossless transmission line has two key properties: the phase velocity v, and the characteristic impedance Z. These are given in terms of the inductance per unit length L, and the capacitance per unit length C:

$$v = \frac{1}{\sqrt{LC}}, \qquad Z = \sqrt{\frac{L}{C}}.$$

The problem, when designing or analysing a transmission line, is to calculate the values of L and C. These are determined by the geometry of the transmission line, and are calculated by solving Maxwell's equations.

As an example, we consider a coaxial cable transmission line, consisting of a central wire of radius a. surrounded by a conducting "sheath" of internal radius d – see Fig. 26.

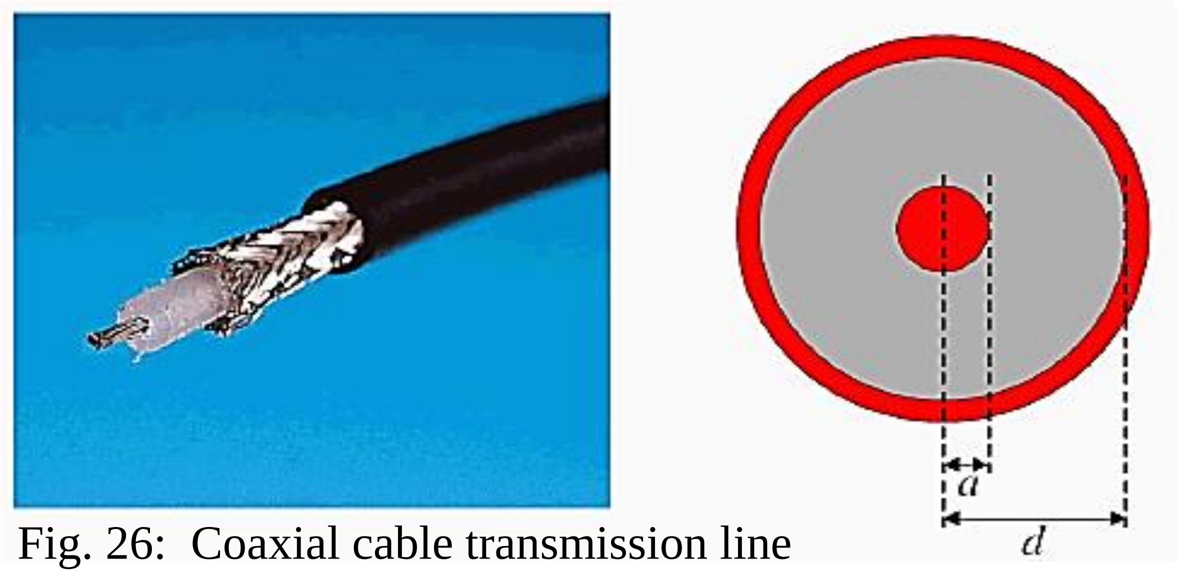
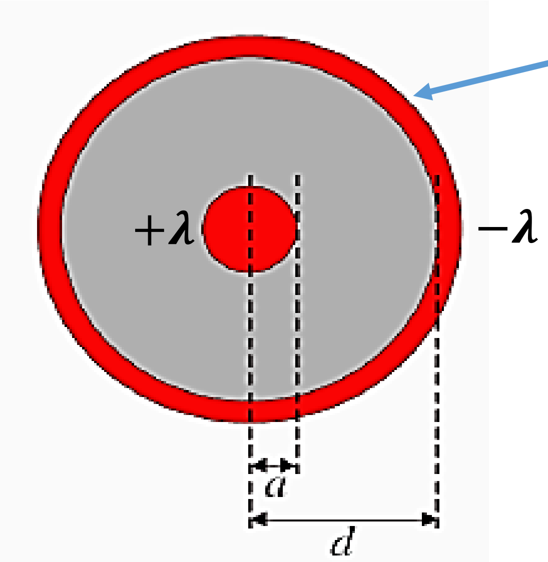


Fig. 26: Coaxial cable transmission line

The center wire and surrounding sheath are separated by a dielectric of permittivity ϵ and permeability μ.

Suppose that the central wire carries charge per unit length $+\lambda$, and the surrounding sheath carries charge per unit length $-\lambda$ (so that the sheath is at zero potential). We can apply Maxwell's equation with Gauss' theorem, to find that the electric field in the dielectric is given by:



$$|\vec{E}| = \frac{\lambda}{2\pi\varepsilon r}, \tag{305}$$

where r is the radial distance from the axis. The potential between the conductors is given by:

$$V = \int_a^d \vec{E} \cdot d\vec{r} = \frac{\lambda}{2\pi\varepsilon} \ln\left(\frac{d}{a}\right). \tag{306}$$

Hence the capacitance per unit length of the coaxial cable is:

$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon}{\ln(d/a)}. \tag{307}$$

To find the inductance per unit length, we consider a length l of the cable. If the central wire carries a current I, then the magnetic field at a radius r from the axis is given by:

$$|\vec{B}| = \frac{\mu I}{2\pi r}. \tag{308}$$

The flux through the shaded area shown in Fig. 27 is given by:

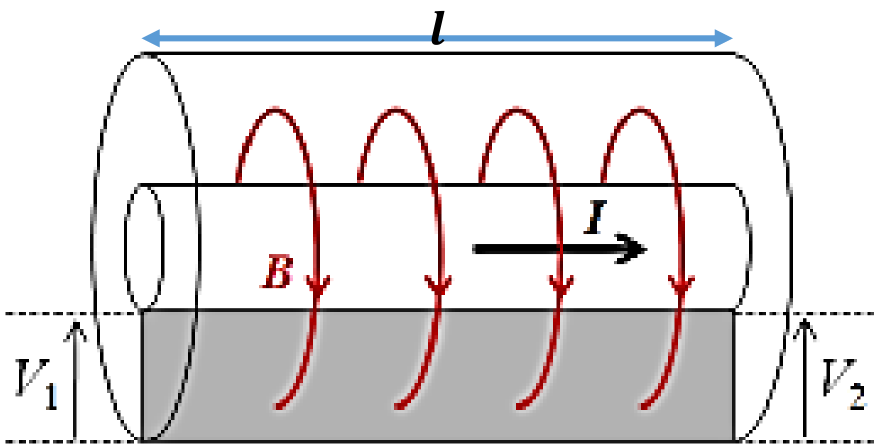
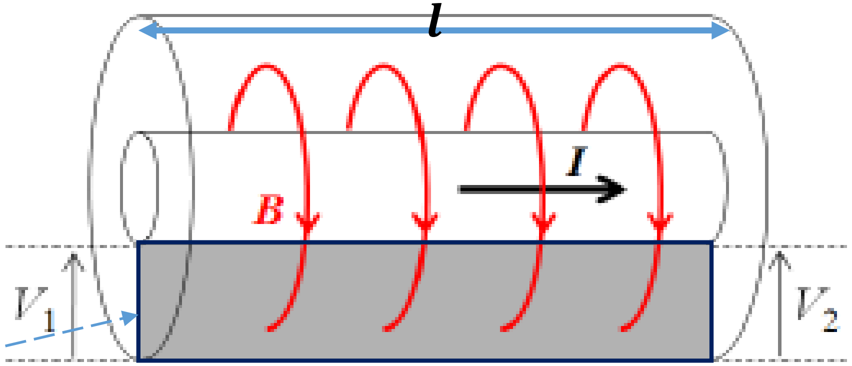


Fig. 27: Inductance in a coaxial cable. A change in the current I flowing in the cable will lead to a change in the magnetic flux through the shaded area; by Faraday's law, the change in flux will induce an electromotive force, which results in a difference between the voltages V_2 and V_1.

$$\Phi = l \int_a^d |\vec{B}| dr = \frac{\mu l I}{2\pi} \ln\left(\frac{d}{a}\right). \tag{309}$$



A change in the flux through the shaded area will (by Faraday's law) induce an electromotive force around the boundary of this area; assuming that the outer sheath of the coaxial cable is earthed, the change in voltage between two points in the cable separated by distance l will be:

$$\Delta V = V_2 - V_1 = -\frac{d\Phi}{dt}. \tag{310}$$

The change in the voltage can also be expressed in terms of the inductance per unit length of the cable:

$$\Delta V = -lL\frac{dI}{dt}, \tag{311}$$

where I is the current flowing in the cable. Hence the inductance per unit length is given by:

$$L = \frac{\Phi}{lI} = \frac{\mu}{2\pi}\ln\left(\frac{d}{a}\right). \tag{312}$$

With the expression in Eq. (307); for the capacitance per unit length one gets:

$$C = \frac{\lambda}{V} = \frac{2\pi\varepsilon}{\ln(d/a)}$$

the phase velocity of waves along the coaxial cable is given by:

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\mu\varepsilon}}, \tag{313}$$

and the characteristic impedance of the cable is given by:

$$Z = \sqrt{\frac{L}{C}} = \frac{1}{2\pi}\ln\left(\frac{d}{a}\right)\sqrt{\frac{\mu}{\varepsilon}}. \tag{314}$$